**Solutions 11 (Ordinary Differential Equations I)**

**1. a)** The Taylor approximation scheme for $k$ = 0, 1, … is:

$$y(x_k+h_k)\approx y(x_k)+\frac{y'(x_k)}{1!}h_k+\frac{y''(x_k)}{2!}h_k^{\;2}+\frac{y'''(x_k)}{3!}h_k^{\;3}+\frac{y^{(4)}(x_k)}{4!}h_k^{\;4}$$

With the formulas for the derivations evolved in the hints we get:

$$x_0=1 \qquad y_0=1$$

$$x_1=1.1 \quad y_1=y_0+\frac{y_0'}{1!}h+\frac{y_0''}{2!}h^2+\frac{y_0'''}{3!}h^3+\frac{y_0^{(4)}}{4!}h^4=1+1h+\frac{2}{3}h^2+\frac{4}{27}h^3+\frac{1}{54}h^4\approx 1.106816666\underline{7}$$

$$x_2=1.2 \quad y_2=y_1+\frac{y_1'}{1!}h+\frac{y_1''}{2!}h^2+\frac{y_1'''}{3!}h^3+\frac{y_1^{(4)}}{4!}h^4\approx\cdots\approx 1.227879708\underline{3}$$

Mind that the derivatives as $y_1'$ are evaluated at $x_1$ and $y_1$ (computed in the step before but not yet rounded !). So the derivatives are approximations, too.

**b)** The globar error is defined as $\underbrace{\max}_{0\le i\le k}\left|y_i-y(x_i)\right|$. More specifically:

$$\underbrace{\max}_{0\le i\le 2}\left|y_i-y(x_i)\right|=\max\left\{\left|y_0-y(x_0)\right|,\left|y_1-y(x_1)\right|,\left|y_2-y(x_2)\right|\right\}=$$

$$\max\left\{\left|1-1\right|,\left|y_1-\left(\frac{1.1^2+2}{3}\right)^{3/2}\right|,\left|y_2-\left(\frac{1.2^2+2}{3}\right)^{3/2}\right|\right\}=1.1473\underline{4}\times 10^{-7}$$

By definition the local (slope) error is:

$$\tau_h(x_n):=\frac{y(x_n+h)-y(x_n)}{h}-\left(\frac{y'(x_n)}{1!}+\frac{y''(x_n)}{2!}h^1+\cdots\frac{y^{(p)}(x_n)}{p!}h^{p-1}\right).$$ For p = 4 together with the derivatives formulas in the hint this is specified to:

$n$ = 0:

$$\tau_h(x_0):=\frac{y(x_0+h)-y(x_0)}{h}-\left(\frac{y'(x_0)}{1!}+\frac{y''(x_0)}{2!}h^1+\cdots+\frac{y^{(4)}(x_0)}{4!}h^{4-1}\right)=$$

$$\frac{y(1.1)-y(1)}{h}-\left(\frac{y'(1)}{1!}+\frac{y''(1)}{2!}h^1+\cdots+\frac{y^{(4)}(1)}{4!}h^{4-1}\right)=$$

$$\frac{\left(\frac{1.1^2+2}{3}\right)^{3/2}-\left(\frac{1^2+2}{3}\right)^{3/2}}{0.1}-\left(1+\frac{2}{3}0.1+\frac{4}{27}0.1^2+\frac{1}{54}0.1^3\right)=\;-6.035828648\times 10^{-7}$$

$$\tau_h(x_1) := \frac{y(x_1+h)-y(x_1)}{h} - \left(\frac{y'(x_1)}{1!} + \frac{y''(x_1)}{2!}h^1 + \cdots + \frac{y^{(4)}(x_1)}{4!}h^{4-1}\right) =$$

*n* = 1:

$$\frac{y(1.2)-y(1.1)}{h} - \left(\frac{y'(1.1)}{1!} + \frac{y''(1.1)}{2!}h^1 + \cdots + \frac{y^{(4)}(1.1)}{4!}h^{4-1}\right) =$$

$$\frac{\left(\frac{1.2^2+2}{3}\right)^{3/2} - \left(\frac{1.1^2+2}{3}\right)^{3/2}}{0.1} - (\ldots) = -5.2265445216\times 10^{-7}$$

Be aware that for the local error the derivatives must be computed using the exact values for $y$. Observe the corresponding notation $y'(x_1)$ a.s.o.

The local errors in $y$ units are $h\cdot\tau_h(x_0) = -5.2265445216\times 10^{-8}$ and $h\cdot\tau_h(x_1) = -6.035828648\times 10^{-8}$.

**2.**

**a)**

$$\varphi_0 = 0 \quad t_0 = 0$$
$$\varphi_1 = \varphi_0 + f(t_0,\varphi_0)h = 0 + c(1-\varepsilon)^2 h = 0.28125$$

**b)**

$$\varphi_0 = 0$$
$$\varphi_{k+1} = \varphi_k + c(1-\varepsilon\cos(\varphi_k))^2 h + \frac{1}{2!}\left(2c\varepsilon\sin(\varphi_k) - c\varepsilon^2\sin(2\varphi_k)\right)c(1-\varepsilon\cos(\varphi_k))^2 h^2 \Rightarrow$$
$$\varphi_1 = \varphi_0 + c(1-\varepsilon\cos(\varphi_0))^2 h + \frac{1}{2!}\underbrace{\left(2c\varepsilon\sin(\varphi_0) - c\varepsilon^2\sin(2\varphi_0)\right)}_{=0}c(1-\varepsilon\cos(\varphi_0))^2 h^2 =$$
$$c(1-\varepsilon)^2 h = 0.28125$$

**c)** The global error in ab) evaluates to | 0.283747 – 0.281250 | = 0.002497

Local error (slope) in a)

$$\tau_h(t_0) := \frac{\varphi(t_0+h)-\varphi(t_0)}{h} - \varphi'(t_0) = \frac{\varphi(0.5)-0}{h} - \varphi'(0) =$$
$$\frac{0.283747-0}{0.5} - c(1-\varepsilon)^2 = 0.004994$$

The local error in b) is the same as in a) becausel $\varphi''(0) = 0$.

The local error in ab) in output units ($y$) is $h\,\tau_h(t_0) = 0.002497$. This is the same as the global error because only the first step was considered.