Solutions 12 (Ordinary Differential Equations II)

1. a) The classical explicit Runge-Kutta scheme of order 4 in classical notation is:

$$k_1 = h\, f(x, y) \qquad k_2 = h\, f(x + \frac{1}{2}h, y + \frac{1}{2}k_1)$$

$$k_3 = h\, f(x + \frac{1}{2}h, y + \frac{1}{2}k_2) \qquad k_4 = h\, f(x + h, y + k_3)$$

$$y(x+h) \approx y(x) + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

Here x, y denote the coordinates already obtained in the preceding step of the scheme, h is the step-size and $f(x, y) = x\, y^{1/3}$ is the r.h.s. function.

The tables below contain the k-values (stages) on the left and the x,y coordinates on the right.

{0.1000000000,	0.1067216175,	0.1068353600,	0.1137855274}
{0.11378488387,	0.1209611687,	0.1210853637,	0.1284998069}
{0.1284990254,	0.1361482228,	0.1362824198,	0.1441778895}
{0.1441769716,	0.15231448320,	0.1524581808,	0.1608487987}
{0.1608477500,	0.1694865529,	0.16963921971,	0.1785368769}

x_k	y_k Classical RK4
1.0000000000	1.0000000000
1.1000000000	1.1068165804
1.2000000000	1.2278795396
1.3000000000	1.3641359063
1.4000000000	1.5165644227
1.5000000000	1.6861704514

b) The globar error is defined as $\underbrace{\max}_{0 \le i \le k} \left| y_i - y(x_i) \right|$. More specifically:

$$\underbrace{\max}_{0 \le i \le 2} \left| y_i - y(x_i) \right| = \max\left\{ \left| y_0 - y(x_0) \right|, \left| y_1 - y(x_1) \right|, \left| y_2 - y(x_2) \right| \right\} =$$

$$\max\left\{ \left|1 - 1\right|, \left| y_1 - \left(\frac{1.1^2 + 2}{3} \right)^{3/2} \right|, \left| y_2 - \left(\frac{1.2^2 + 2}{3} \right)^{3/2} \right| \right\} = 5.392586333 \times 10^{-8}$$

By definition the local (slope) error in classical notation is:

$$\tau_h(x_n) := \frac{y(x_n + h) - y(x_n)}{h} - \left(\frac{\tilde{k}_1 + 2\tilde{k}_2 + 2\tilde{k}_3 + \tilde{k}_4}{6h} \right).$$

The tilde superscript □ indicates that the stage values k must be evaluated using exact values for x, y (!). These are listed in the table below. Observe that there is no difference in the very first step since initially x and y are exact.

{0.1000000000,	0.1067216175,	0.1068353600,	0.1137855274}
{0.11378488476,	0.12096116960,	0.1210853646,	0.1284998078}

$n = 0$:

$$\tau_h(x_0) := \frac{y(x_0+h)-y(x_0)}{h} - \left(\frac{\tilde{k}_1+2\tilde{k}_2+2\tilde{k}_3+\tilde{k}_4}{6h}\right) =$$
$$\frac{\left(\frac{1.1^2+2}{3}\right)^{3/2} - \left(\frac{1^2+2}{3}\right)^{3/2}}{0.1} - \left(\frac{\tilde{k}_1+2\tilde{k}_2+2\tilde{k}_3+\tilde{k}_4}{0.6}\right) = 2.5922467994\times10^{-7}$$

$n = 1$:

$$\tau_h(x_1) := \frac{y(x_1+h)-y(x_1)}{h} - \left(\frac{\tilde{k}_1+2\tilde{k}_2+2\tilde{k}_3+\tilde{k}_4}{6h}\right) =$$
$$\frac{\left(\frac{1.2^2+2}{3}\right)^{3/2} - \left(\frac{1.1^2+2}{3}\right)^{3/2}}{0.1} - \left(\frac{\tilde{k}_1+2\tilde{k}_2+2\tilde{k}_3+\tilde{k}_4}{0.6}\right) = 2.7090772847\times10^{-7}$$

The local errors in y units then are $h\cdot\tau_h(x_0) = 2.5922467994\times10^{-8}$ and $h\cdot\tau_h(x_1) = 2.7090772847\times10^{-8}$.

The results on the errors are compatible with Exercises 11, P1. The orders of magnitudes are even the same.

2. Computations and evaluations of derivatives (up to order p) of the r.h.s. function f are avoided.

3. The scheme for the very first step is laid out on the right. The rounded k-values evaluate to

{ 0.2812500000,
0.2831039348,
0.2831284565,
0.2887646287 }.

The absolute global error evaluates to 2.9540931684×10⁻⁷

$$\left.\begin{aligned} k_1 &= h\,c\left(1-\varepsilon\cos(\phi_0)\right)^2 \\ k_2 &= h\,c\left(1-\varepsilon\cos(\phi_0+\tfrac{1}{2}k_1)\right)^2 \\ k_3 &= h\,c\left(1-\varepsilon\cos(\phi_0+\tfrac{1}{2}k_2)\right)^2 \\ k_4 &= h\,c\left(1-\varepsilon\cos(\phi_0+k_3)\right)^2 \end{aligned}\right\}\ 4\ stages$$

$$\phi_0 = \phi(t_0) = \phi(0) = 0$$
$$\phi_1 = \phi_0 + \frac{1}{6}\left(k_1+2k_2+2k_3+k_4\right) = 0.283746568\underline{6}$$

The Local error (slope)

$$\tau_h(t_0) := \frac{\phi(t_0+h)-\phi(t_0)}{h} - \frac{1}{6h}\left(k_1+2k_2+2k_3+k_4\right) = \frac{\phi(0.5)-0}{0.5} - \frac{1}{6\cdot0.5}\left(k_1+2k_2+2k_3+k_4\right)$$
$$= 5.9081863368\times10^{-7}$$

The local error in output units (y) is $h\,\tau_h(t_0) = 2.9540931684 \times 10^{-7}$. This must be the same as the global error because only the first step was considered.

4. The stage value k_s at the very end of a computational step in the Runge-Kutta scheme is the same as the first stage value k_1 at the very start of the succeeding step. This saves at least one evaluation of the r.h.s. function f per step.

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5. Cf. separate file (.pdf)