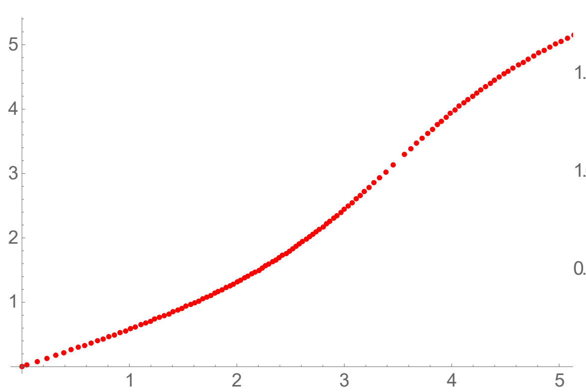
Solutions 13 (Ordinary Differential Equations II / III)

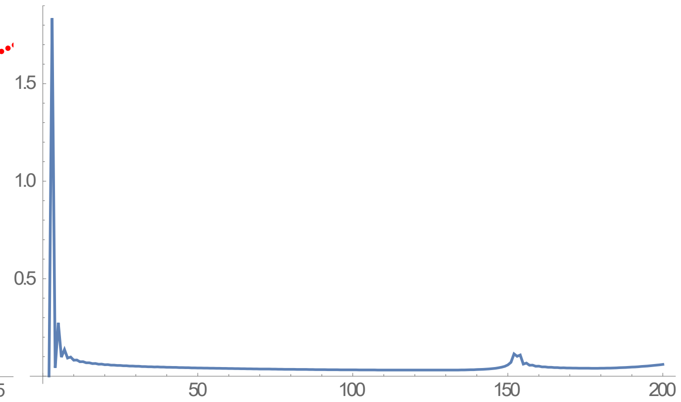
1. From the script we get that the estimation of the error is equal to $e_k = \frac{1}{2} h_k (k_2 - k_1)$ and the step-size adaptation is ruled by the formula $h_{new} = h_k \sqrt[2]{\left(\frac{\varepsilon}{|e_k|}\right)} = h_k \left(\frac{|e_k|}{\varepsilon}\right)^{-1/2}$ and $\varepsilon = 10^{-4} + 10^{-4} |y_k|$.

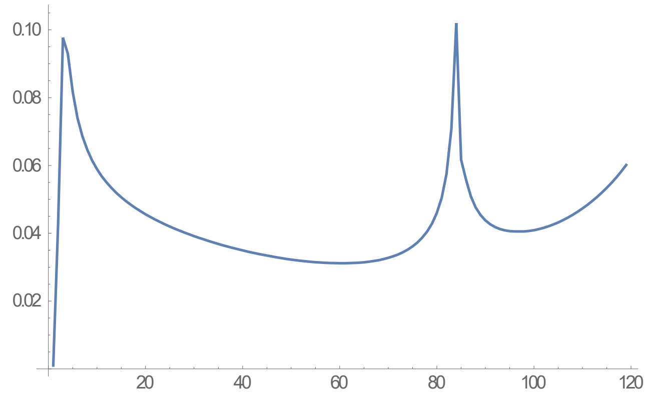
TableForm=

x	y	h	e_k	$\left(\left|\frac{e_k}{e_a + y\, e_r}\right|\right)^{-1/2}$	h_{new}	Status	k1	k2
0	0	0.001000000000	$2.966308593 \times 10^{-11}$	1836.080994	1.836080994	Proceed	0.5625000000	0.5625000593
0.001000000000	0.0005625000297	1.836080994	0.1816906908	0.02346690271	0.04308713406	Reject	0.5625000593	0.7604114711
0.001000000000	0.0005625000297	0.04308713406	$2.482928795 \times 10^{-6}$	6.348044733	0.2735190544	Proceed	0.5625000593	0.5626153108
0.04408713406	0.02480149842	0.2735190544	0.0008029601142	0.3572505363	0.09771482888	Reject	0.5626153339	0.5684866625
0.04408713406	0.02480149842	0.09771482888	0.00005266807814	1.394909848	0.1363033771	Proceed	0.5626153339	0.5636933295
0.1418019629	0.07983002758	0.1363033771	0.0002321903226	0.6819548817	0.09295275337	Reject	0.5636949057	0.5671018693
0.1418019629	0.07983002758	0.09295275337	0.00009682564780	1.056045170	0.09816230619	Proceed	0.5636949057	0.5657782361
0.2347547163	0.1323238468	0.09816230619	0.0001636450839	0.8318285944	0.08165421318	Reject	0.5657830389	0.5691172125
0.2347547163	0.1323238468	0.08165421318	0.0001099296895	1.014910688	0.08287173370	Proceed	0.5657830389	0.5684756052

The next plots show the coordinates of the solution and the evolution of total (200) and proceeded (119) step-sizes.







2. We apply the "midpoint method" to the Dahlquist equation $y' = Ay \quad y(0) = 1 \ (A \in \mathbb{C})$. This implies the r.h.s. formula $f(x, y) = Ay$ and

$$y_0 = 1$$
$$y_{k+1} = y_k + h\left(Ay_k + A\left(\frac{1}{2}hAy_k\right)\right) = \underbrace{\left(1 + hA + \frac{1}{2}(hA)^2\right)}_{Function\ F(hA)=F(z)} y_k$$

Therefore the A-stability polynomial is $F(z) = 1 + z + \frac{z^2}{2} \quad (z = hA)$ .This is the same polynomial as in Example 1.11 (Heun method) and from there we get (-2,0) for the A-stability interval on the negative real axis.

3. The A-stability polynomial is

$$F(z) = 1 + b_1 k_1(z) + b_2 k_2(z) + b_3 k_3(z) + 0 k_4(z)$$

whereas

$$k_1(z) = z$$
$$k_2(z) = z\left(1 + \frac{1}{2}k_1(z)\right) = z\left(1 + \frac{1}{2}z\right)$$
$$k_3(z) = z(1 + -1k_1(z) + 2k_2(z)) = z\left(1 - z + 2z\left(1 + \frac{1}{2}z\right)\right) = z + z^2 + z^3$$

Therefore $F(z) = 1 + \frac{1}{6}k_1(z) + \frac{2}{3}k_2(z) + \frac{1}{6}k_3(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3$.
Cf. Example 1.13 (script).

4. The A-stability polynomial is the same as in Problem 2 ("midpoint method"):

$$F(z) = 1 + z + \frac{z^2}{2} \quad (z = hA).$$

The A-stability interval on the negative real axis therefore is (-2, 0). Substituting A with the partial derivative $f_y = -\frac{2y}{2\sqrt{x^2 + y^2}}$ yielding the value -1 at (0,4) … the stiffness test reduces to the A-stability constraint $-2 < Ah < 0 \Leftrightarrow -2 < f_y h < 0 \Leftrightarrow -2 < -h < 0$, and this is fulfilled for h = 1. It can be concluded that the stiffness test is negative (no stiffness detected).