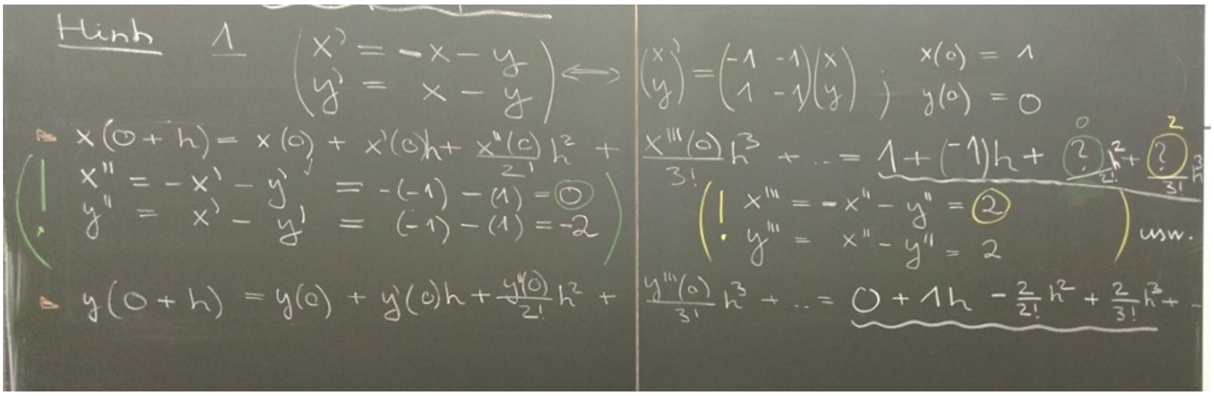
Solutions 14: Numeric Ordinary Differential Equations IV

1. Solution from the classroom:



2. Cf. P20.6 from Schaum's Outline of Numerical Analysis, Chapter 20.

Let $x' = u$, $y' = v$, $z' = w$ be the velocity components. Then

$$u' = f_1(t, x, y, z, u, v, w) \qquad v' = f_2(t, x, y, z, u, v, w) \qquad w' = f_3(t, x, y, z, u, v, w)$$

These six equations are the required first-order system. Other systems of higher-order equations may be treated in the same way.

The dimension of the first-order ODE-system is 6.

3. Cf. P20.4 from Schaum's Outline of Numerical Analysis, Chapter 20.

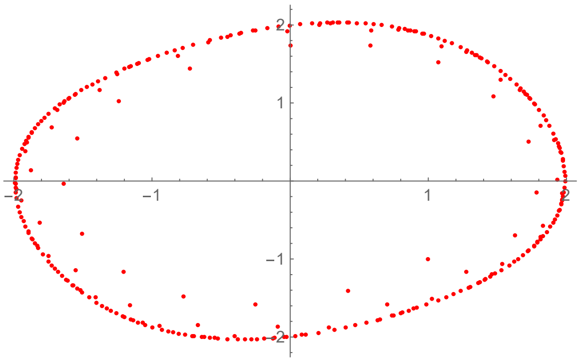
Let the second-order equation be $y'' = f(x, y, y')$. Then introducing $p = y'$ we have at once $y' = p$, $p' = f(x, y, p)$. As a result of this standard procedure a second-order equation may be treated by system methods if this seems desirable.

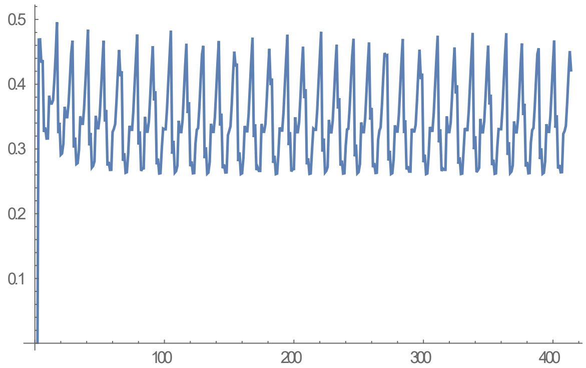
The initial conditions fort he vector (p, y) are: $p(x_0) = y_1$, $y(x_0) = y_0$.

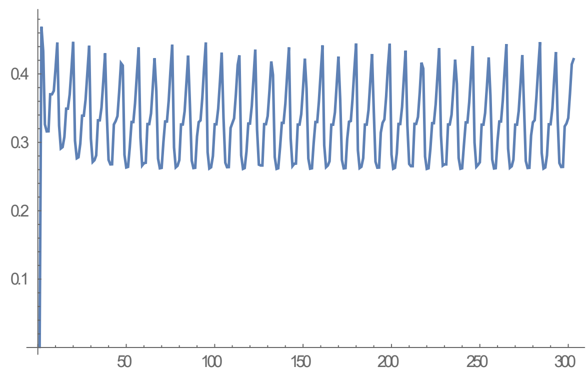
4. From the script we know that the estimation of the error is equal to $\vec{e}_k = \frac{1}{2} h_k (\vec{k}_2 - \vec{k}_1)$ and the step-size adaptation is ruled by the formula $h_{new} = h_k \left\| \frac{\vec{e}_n}{\varepsilon_a + \varepsilon_r \vec{y}_n} \right\|^{-1/2}$ with $\varepsilon_a = 10^{-1}, \varepsilon_r = 10^{-2}$.

x	y	h	e_k	$\left\Vert \frac{e_k}{\varepsilon_a + Y \varepsilon_r} \right\Vert$	h_{neu}	Status	k1
0	(1.000000000, -1.000000000)	0.001000000000	($-5.000000000 \times 10^{-7}$, $2.999001000 \times 10^{-7}$)	469.0415760	0.4690415760	Proceed	(-1.00, -1.00
0.001000000000	(0.9989995000, -1.000999700)	0.4690415760	(-0.1099339890, 0.06059334319)	1.000254693	0.4691610377	Proceed	(-1.0, -0.99
0.4700415760	(0.4195550341, -1.409166461)	0.4691610377	(-0.07173212952, 0.1337985989)	0.9234240375	0.4332345797	Reject	(-1.4, -0.65
0.4700415760	(0.4195550341, -1.409166461)	0.4332345797	(-0.06116684426, 0.1119369002)	1.009579027	0.4373845456	Proceed	(-1.4, -0.65
0.9032761557	(-0.2521114495, -1.579602423)	0.4373845456	(-0.004182800260, 0.2080474579)	0.7460460194	0.3263089992	Reject	(-1., -0.04
0.9032761557	(-0.2521114495, -1.579602423)	0.3263089992	(-0.002328082473, 0.1109948425)	1.021398991	0.3332916827	Proceed	(-1., -0.04
1.229585155	(-0.7698780179, -1.482876770)	0.3332916827	(0.03605134912, 0.1277328531)	0.9481434422	0.3160083232	Reject	(-1.48, 0.649

The next plots show the vector coordinates $\{z, v\}$ of the solution evolving in time (100 sec) and the evolution of total (414) and proceeded (303) step-sizes.







5. Cf. P20.2 from Schaum's Outline of Numerical Analysis, Chapter 20.

$$k_1 = hf_1(x_n, y_n, p_n) \qquad k_3 = hf_1(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}k_2,\ p_n + \tfrac{1}{2}l_2)$$
$$l_1 = hf_2(x_n, y_n, p_n) \qquad l_3 = hf_2(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}k_2,\ p_n + \tfrac{1}{2}l_2)$$
$$k_2 = hf_1(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}k_1,\ p_n + \tfrac{1}{2}l_1) \qquad k_4 = hf_1(x_n + h,\ y_n + k_3,\ p_n + l_3)$$
$$l_2 = hf_2(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}k_1,\ p_n + \tfrac{1}{2}l_1) \qquad l_4 = hf_2(x_n + h,\ y_n + k_3,\ p_n + l_3)$$
$$y_{n+1} = y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$p_{n+1} = p_n + \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

6. Cf. P20.7 from Schaum's Outline of Numerical Analysis, Chapter 20.

An equivalent first-order system is

$$y' = p = f_1(t, y, p)$$
$$p' = -y + (.1)(1 - y^2)p = f_2(t, y, p)$$

The Runge–Kutta formulas for this system are

$$k_1 = hp_n \qquad l_1 = h[-y_n + (.1)(1 - y_n^2)p_n]$$
$$k_2 = h\left(p_n + \frac{1}{2}l_1\right) \qquad l_2 = h\left\{-\left(y_n + \frac{1}{2}k_1\right) + (.1)\left[1 - \left(y_n + \frac{1}{2}k_1\right)^2\right]\left(p_n + \frac{1}{2}l_1\right)\right\}$$
$$k_3 = h\left(p_n + \frac{1}{2}l_2\right) \qquad l_3 = h\left\{-\left(y_n + \frac{1}{2}k_2\right) + (.1)\left[1 - \left(y_n + \frac{1}{2}k_2\right)^2\right]\left[p_n + \frac{1}{2}l_2\right)\right\}$$
$$k_4 = h(p_n + l_3) \qquad l_4 = h\{-(y_n + k_3) + (.1)[1 - (y_n + k_3)^2](p_n + l_3)\}$$

and $$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad p_{n+1} = p_n + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

Choosing $h = .2$, computations produce the following results to three places:

$$k_1 = (.2)(0) = 0 \qquad l_1 = (.2)[-1 + (.1)(1-1)(0)] = -.2$$
$$k_2 = (.2)(-.1) = -.02 \qquad l_2 = (.2)[-1 + (.1)(1-1)(-.1)] = -.2$$
$$k_3 = (.2)(-.1) = -.02 \qquad l_3 = (.2)[-.99 + (.1)(.02)(-.1)] \simeq -.198$$
$$k_4 \simeq (.2)(-.198) \simeq -.04 \qquad l_4 = (.2)[-(.98) + (.1)(.04)(-.198)] \simeq -.196$$

These values now combine into

$$y_1 \simeq 1 + \frac{1}{6}(-.04 - .04 - .04) = .98$$

$$p_1 \simeq 0 + \frac{1}{6}(-.2 - .4 - .396 - .196) \simeq -.199$$