Solutions 1

1 (8.6)

x	y	I	II	III
0	1			
1	1	0		
2	2	1	1/2	
4	5	3/2	1/6	$-\frac{1}{12}$

(order)

Ex. $-\frac{1}{12} = y(0,1,2,4) = \frac{y(1,2,4) - y(0,1,2)}{4-0} = \frac{1/6 - 1/2}{4}$

$\frac{1}{6} = y(1,2,4) = \frac{y(2,4) - y(1,2)}{4-1} = \frac{3/2 - 1}{3}$

2 (8.7)

Directly $y(x_0, x_1) = \frac{y(x_1) - y(x_0)}{x_1 - x_0}$

Alg. $\frac{y(x_0) - y(x_1)}{x_0 - x_1} = y(x_1, x_0)$ ☑

3 (8.8) Direct computation is tedious, but there is an elegant logical reasoning!

Consider the data:

x	x_0	x_1	x_2
y	y_0	y_1	y_2

Interpolating these 3 points results in a parabola $p = a_0 \underbrace{\Pi_0}_{1} + a_1 \underbrace{(x - x_0)}_{\Pi_1} + a_2 \underbrace{(x - x_0)(x - x_1)}_{\Pi_2}$. The coefficient of x^2 is $a_2 = y(x_0, x_1, x_2)$. Changing the order of the data does not change the parabola and thus does not change the number before x^2 ☑

4 a)

x	y	I	II	III
0	1			
1	−1	−2		
4	1	2/3	2/3	
6	−1	−1	−1/3	$-\frac{1}{6}$

(8.23)

x	y	I	II	III
4	1			
1	−1	2/3		
6	−1	0	−1/3	
0	1	−1/3	1/3	$-\frac{1}{6}$

(8.25) ✓

(Sol. 1 cont.)

5 (8.29) Cf. the hint! $y(x) = \pi_{n+1}(x)$ and $y(x)$ is 0 for $x = x_k$ $(k = 0, \dots, n)$

a) (1) By induction: $y(x_k) = 0$ $(k = 0, \dots, n)$ $\Rightarrow$
$y(x_0, x_1) = 0$ and $y(x_1, x_2) = 0$ $\Rightarrow$ $y(x_0, x_1, x_2) = 0$
a.s.o.

Divided differences of order 0 are 0 $\Rightarrow$
" " " " 1 " 0 $\Rightarrow$
" " " " 2 " 0 $\Rightarrow$ ...

(2) By logics (!): y is a polynomial and its Newton interpolation obviously is $y = \pi_{n+1} \stackrel{!}{=}$
$= 0 \cdot \pi_0 + 0 \cdot \pi_1 + 0 \cdot \pi_2 + 0 \cdot \pi_3 + \dots + 0 \cdot \pi_n + 1\, \pi_{n+1}$.
Thus all coefficient up to π_n are zero, but these are $y(x_0)$, $y(x_0, x_1)$, $y(x_0, x_1, x_2)$, ..., $y(x_0, x_1, x_2, \dots, x_n)$ □

b) (1) Firstly consider x as a further argument x_{n+1} different from any of $x_0, x_1, \dots, x_n$.
Then $y(x_0, x_1, \dots, x_n, x_{n+1})$ is the coefficient of π_{n+1} and must be 1 by a) (2) □

(2) By theory we have that

$$\underbrace{y(x_0, x_1, \dots, x_n, x)}_{\text{"}n+2\text{"}} \stackrel{!}{=} \frac{y^{(n+1)}(\xi)}{(n+1)!}$$

$\xi \in$ (minimal x, maximal x).

But $y^{(n+1)}(x) = (n+1)!$ since y is a polynomial of the form $1 \cdot x^{n+1} + \dots$ □

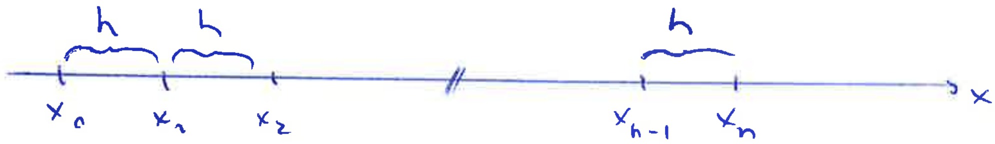
c) As in b)(1) or as in b)(2):

(Sol. 1 cont.)

5 (cont.) c) $$\underbrace{y(x_0, \dots, x_n, x, z)}_{\text{"}n+3\text{"}} = \frac{y^{(n+2)}(\xi)}{(n+2)!}$$ but

the derivative $y^{(n+2)}(\xi)$ is 0 because y has polynomial degree $n+1$ $\square$

6 (8.31)



$h = \Delta x$ = constant, or $x_k = x_0 + k \cdot h$ $(k = 0, \dots, n+1)$

Explanation of the Δ operator

Δ^0	$y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad \dots$
$\Delta^1 = \Delta$	$y_1 - y_0 = \Delta y_0 \quad y_2 - y_1 = \Delta y_1 \quad y_3 - y_2 = \Delta y_2 \quad y_4 - y_3 = \Delta y_3 \quad \dots$
Δ^2	$\Delta y_1 - \Delta y_0 = \Delta^2 y_0 \quad \Delta y_2 - \Delta y_1 = \Delta^2 y_1 \quad \Delta y_3 - \Delta y_2 = \Delta^2 y_2 \quad \dots$
$\vdots$	
Δ^n	$\Delta^{n-1} y_1 - \Delta^{n-1} y_0 = \Delta^n y_0 \quad \Delta^{n-1} y_2 - \Delta^{n-1} y_1 = \Delta^n y_1 \quad \dots$

Induction

$k = 0$: $y(x_0) = y_0 = \Delta^0 y_0 / (0!\, h^0) = y_0$ ✓

$k = 1$: $y(x_0, x_1) = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{\Delta^1 y_0}{h} = \dfrac{\Delta^1 y_0}{h^1\, 1!}$ ✓

$k = 2$: $$y(x_0, x_1, x_2) \overset{!}{=} \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0} \overset{!}{=}$$

$$= \frac{\frac{\Delta^1 y_1}{h} - \frac{\Delta^1 y_0}{h}}{2h} = \frac{\Delta^2 y_0}{2 h^2}$$ ✓

$k = 3$: $$y(x_0, x_1, x_2, x_3) = \frac{y(x_1, x_2, x_3) - y(x_0, x_1, x_2)}{x_3 - x_0} = \dots$$

(Sol. 1 cont.)

6. (cont.)

$$\ldots = \frac{\dfrac{\Delta^2 y_1}{2h^2} - \dfrac{\Delta^2 y_0}{2h^2}}{3h} = \frac{\Delta^3 y_0}{3\cdot 2\cdot h^3} = \frac{\Delta^3 y_0}{h^3\cdot 3!} \quad \checkmark$$

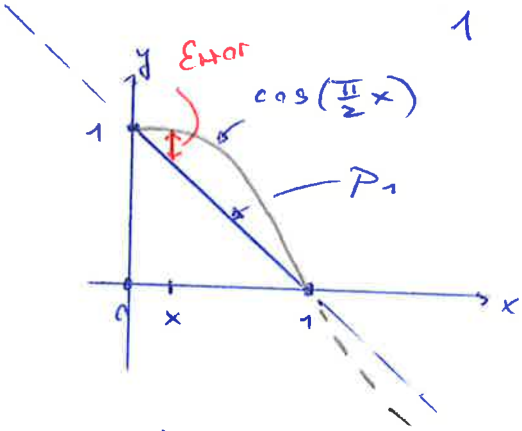
a.s.o.

7. (2.9.) Data

x	y	
0	1	
1	0	-1

$$p_1 = 1\cdot 1 + (-1)(x-0) = 1 - x$$

(linear)



By theory

$$y - p_1 = \frac{y''(\xi)}{2!}\pi_2 = \frac{-\left(\frac{\pi}{2}\right)^2 \cos\left(\frac{\pi}{2}\xi\right)}{2!}(x)(x-1)$$

$(\xi \in (0,1))$

Since $|\cos(\ldots)| \le 1$ we get the estimation:

$$|y - p_1| \le \frac{\pi^2}{4}\cdot\frac{1}{2!}|x(x-1)| = \frac{\pi^2}{8}|x^2 - x|$$

Check at $x = \frac{1}{2}$: $|y - p_1| \le \frac{\pi^2}{8}\cdot\frac{1}{4} = \frac{\pi^2}{32} \approx 0.308425$

The exact error is

$$\left|\cos\left(\frac{\pi}{4}\right) - \frac{1}{2}\right| = \left|\frac{\sqrt{2}}{2} - \frac{1}{2}\right| \approx 0.207107$$

8 (2.14 - 2.18) Data

x	y	I	II
0	0		
1	1	1	
2	0	-1	-1

$$p_2 = 0\cdot 1 + 1(x-0) + (-1)(x-1)x = -x^2 + 2x$$

(quadratic)

a) Error $= y(x) - p_2(x) = \left(\sin\left(\frac{\pi}{2}x\right) - (-x^2+2x)\right) \stackrel{!}{=}$

$$= \frac{y^{(3)}(\xi)}{3!}(x-0)(x-1)(x-2)\,; \quad x, \xi \in (0,2)$$

(Sol. 1 cont.)

8 (cont.) b) $g^{(3)}(\xi) = -\frac{\pi^3}{8} \cdot \cos\left(\frac{\pi}{2}\xi\right)$ and thus

$$|Error| \le \frac{\pi^3}{8 \cdot 3!} |x(x-1)(x-2)| \quad (x \in [0,2])$$

b) (cont.) Check at $x = \frac{1}{2}$: $|Error| \le \frac{\pi^3}{48} \left|\frac{1}{2} \cdot \left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)\right|$

≈ 0.242237

The exact error is:

$|Error| = \left|\sin\left(\frac{\pi}{4}\right) - P_2\left(\frac{1}{2}\right)\right| = \left|\frac{\sqrt{2}}{2} - \frac{3}{4}\right|$

≈ 0.042893

c) $g'(x) = \frac{\pi}{2} \cos\left(\frac{\pi}{2}x\right) \Rightarrow g'\left(\frac{1}{2}\right) = \frac{\pi}{2} \cdot \frac{\sqrt{2}}{2} \approx 1.110721$

$P_2'(x) = -2x + 2 \Rightarrow p'\left(\frac{1}{2}\right) = 1$

d) $g''(x) = -\frac{\pi^2}{4} \sin\left(\frac{\pi}{2}x\right) \Rightarrow g''\left(\frac{1}{2}\right) = -\frac{\pi^2}{4} \cdot \frac{\sqrt{2}}{2} \approx -1.744716$

$P_2''(x) = -2$

e) $\int_0^2 \sin\left(\frac{\pi}{2}x\right)dx = -\frac{2}{\pi} \cdot \cos\left(\frac{\pi}{2}x\right)\Big|_0^2 = -\frac{2}{\pi}(-1-1) = \frac{4}{\pi}$

≈ 1.273240

$\int_0^2 (-x^2 + 2x)\,dx = \left(\frac{-x^3}{3} + x^2\right)\Big|_0^2 = \frac{-8}{3} + 4 = \frac{4}{3} \approx 1.333333$

Typical phenomenon: Low degree polynomial interpolation is not adequate for derivatives computations but it is rather suitable for integral computations.

(Sol. 1 cont.)

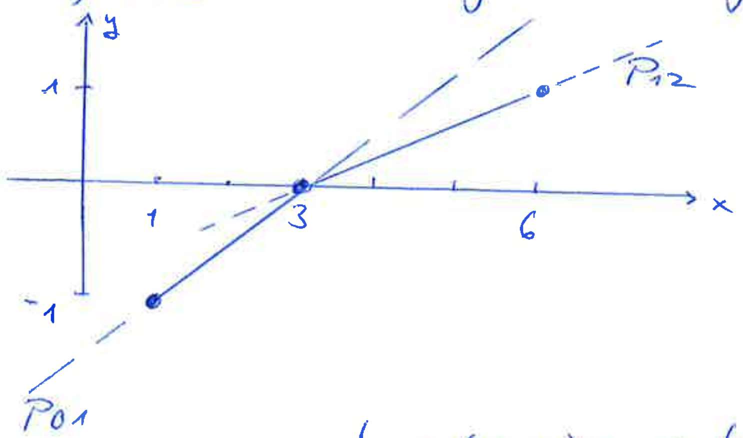
9

x	y	
1	-1	$= p_0$
3	0	$= p_1$
6	1	$= p_2$

$$p_{01} = \frac{-(x-x_1)p_0 + (x-x_0)p_1}{x_1 - x_0} = \overbrace{\frac{-(x-3)(-1)+(x-1)\cdot 0}{2}}^{\frac{-3+x}{2}} \checkmark$$

$$p_{12} = \frac{-(x-6)p_1^{0} + (x-3)\cdot 1}{(x_2 - x_1 = 6-3=3)} = \frac{x-3}{3} \checkmark$$

p_{01} ; p_{12} are straight lines joining neighboured pts.



Finally, $p_{012} \overset{!}{=} \dfrac{-(x-x_2)p_{01} + (x-x_0)p_{12}}{x_2 - x_0} = \ldots$

$$= \underline{\frac{-(x-6)\left(\frac{x-3}{2}\right) + (x-1)\left(\frac{x-3}{3}\right)}{5}} \quad \text{(parabola)}$$

(Check: $p_{012}(1) = -1 \checkmark$; $p_{012}(3) = 0 \checkmark$; $p_{012}(6) = 1 \checkmark$)