Solutions 2

1 (10.3) Scheme

cubic degree

	x	y	I	II	III
	0	0			
(!)	0	0	0		
	4	2	$\boxed{2/4}$	$\boxed{\frac{1}{8}}$	
(!)	4	2	0	$\boxed{-\frac{1}{8}}$	$\boxed{-\frac{1}{16}}$

$$\overset{!}{\Longrightarrow} \quad P_3(x) = 0 \cdot 1 + 0\,(x-0) + \frac{1}{8}(x-0)^2 - \frac{1}{16}(x-0)^2(x-4)$$

$$= -\frac{1}{16}x^2(x-6) = -\frac{1}{16}x^3 + \frac{3}{8}x^2$$

$(x \in [0,4])$

2 (10.6) Scheme

x	y	I	II	III	IV	V
x_0	y_0					
x_0	y_0	y_0'				
x_0	y_0	y_0'	$\frac{1}{2}y_0''$			
x_1	y_1	$\boxed{A}$	$\boxed{B}$	$\boxed{D}$		
x_1	y_1	y_1'	$\boxed{C}$	$\boxed{E}$	$\boxed{G}$	
x_1	y_1	y_1'	$\frac{1}{2}y_1''$	$\boxed{F}$	$\boxed{H}$	$\boxed{I}$

- P has degree 5
- $$P = y_0 \pi_0 + y_0' \underbrace{(x-x_0)}_{\pi_1} + \frac{1}{2}y_0''(x-x_0)^2 + D\,(x-x_0)^3$$
$$+ G\,(x-x_0)^3(x-x_1) + I\,(x-x_0)^3(x-x_1)^2$$

$A, B, C \ldots H, I$

E.g. $A = \dfrac{y_1 - y_0}{x_1 - x_0}$; $B = \dfrac{A - y_0'}{x_1 - x_0}$; $D = \dfrac{B - \frac{1}{2}y_0''}{x_1 - x_0}$
$= y(x_0, x_0, x_0, x_1)$;

$C = \dfrac{y_1' - A}{x_1 - x_0}$; $E = \dfrac{C - B}{x_1 - x_0}$; $F = \dfrac{\frac{1}{2}y_1'' - C}{x_1 - x_0}$;

$G = \dfrac{E - D}{x_1 - x_0} = y(x_0, x_0, x_0, x_1, x_1)$; $H = \dfrac{F - E}{x_1 - x_0}$

$I = \dfrac{H - G}{x_1 - x_0} = y(x_0, x_0, x_0, x_1, x_1, x_1)$.

(Sol. 2 cont.)

3 (10, 10) (cf. 2)

Scheme (degree 5)

x	y	I	II	III	IV	V
0	0					
0	0	0				
0	0	0	0			
1	1	1	1	1		
1	1	1	0	-1	-2	
1	1	1	0	0	1	3

$$\Longrightarrow P_5 = (0+0+0) + 1\,x^3 + (-2)\,x^3(x-1) + 3\,x^3(x-1)^2$$

$$= 3x^5 - 8x^4 + 6x^3$$

4 (10, 12) $a = P_1'(2)$ (!)

Scheme

x	y	I	II	III	IV	V
0	0					
0	0	0				
0	0	0	0			
2	1	1/2	1/4	1/8		
2	1	a	A	B	D	
2	1	a	0	C	E	F

$$A = \frac{a-\frac{1}{2}}{2-0} = \frac{2a-1}{4}$$

$$B = \frac{A-\frac{1}{4}}{2-0} = \frac{2a-2}{8} = \frac{a-1}{4}$$

$$D = \frac{B-\frac{1}{8}}{2-0} = \frac{2a-3}{16}$$

$$C = \frac{0-A}{2-0} = \frac{1-2a}{8}$$

$$E = \frac{C-B}{2-0} = \frac{3-4a}{16}$$

$$F = \frac{E-D}{2-0} = \frac{3-3a}{16}$$

As P_1 must be of 4th degree, we conclude that $F = 0$ (!) and thus $3-3a = 0 \Longleftrightarrow a = 1$

$$\Longrightarrow P_1 = 0+0+0+\frac{1}{8}x^3 - \frac{1}{16}x^3(x-2) = -\frac{1}{16}x^4 + \frac{1}{4}x^3$$

$(0 \le x \le 2)$

(Sol. 2 cont.)

4 (cont.)

Scheme for P_2

$P_2'(2) = a = 1$ (!)

x	y	I	II	III	IV	V
2	1					
2	1	1				
2	1	1	0			
4	2	1/2	-1/4	-1/8		
4	2	0	-1/4	0	1/16	
4	2	0	0	1/8	1/16	0 (!)

P_2 is of 4th degree ✓ (!)

$$P_2 = 1 + 1(x-2) + 0(x-2)^2 - \frac{1}{8}(x-2)^3 + \frac{1}{16}(x-2)^3(x-4)$$

$$= \frac{1}{16}x^4 - \frac{3x^3}{4} + 3x^2 - 4x + 2 \qquad (2 \le x \le 4)$$

5 (10.15–10.16) Schemes

a)

x	y	I	II	III	IV
0	1				
0	1	0			
1	0	-1	-1		
2	9	9	5	3	
2	9	24	15	5	1

(!)

$$P_{a)} = 1 - 1x^2 + 3x^2(x-1) + 1x^2(x-1)(x-2)$$

$$= x^4 - 2x^2 + 1 \qquad (x \in [0,2])$$

b)

x	y	I	II	III	IV
0	1				
0	1	-1			
0	1	-1	0		
1	2	1	2	2	
1	2	7	6	4	2

$$P_{b)} = 1 - 1x + 2x^3 + 2x^3(x-1)$$

$$= 2x^4 - x + 1 \qquad (x \in [0,1])$$

6

a) $\dfrac{y^{(n+1)}(\xi)}{(n+1)!}$

$\xi \in$ (minimal x, maximal x)

b) $\dfrac{y^{(n)}(\xi)}{n!}$

c) $\dfrac{y^{(n)}(x_0)}{n!}$ (!)

(Sol. 2 cont.)

7. $y = \cos\frac{\pi}{2}x \Rightarrow y' = -\frac{\pi}{2}\sin\frac{\pi}{2}x = \begin{cases} 0 & (x=0) \\ -\frac{\pi}{2} & (x=1) \end{cases}$

Newton osculation

x	y	y'		
0	1			
0	1	0		
1	0	$\lvert -1 \rvert$	-1	
1	0	$-\frac{\pi}{2}$	$\lvert -\frac{\pi}{2}+1 \rvert$	$-\frac{\pi}{2}+2$

$$p = 1\cdot 1 + 0(x-0) + (-1)(x-0)^2 + \left(2-\frac{\pi}{2}\right)x^2(x-1)$$

(cubic)

$$\underline{\text{Error}} = (y-p)(x) \doteq \frac{y^{(4)}(\xi)}{4!}x^2(x-1)^2 \quad (\xi \in (0;1))$$

$$|\text{Error}| \le \underbrace{\max_{x\in[0;1]}|y^{(4)}(x)|}_{\le \left(\frac{\pi}{2}\right)^4\cdot 1}\ \Big/\ \frac{1}{4!}\underbrace{x^2(x-1)^2}_{\text{max at } x=\frac{1}{2}} \le \left(\frac{\pi}{2}\right)^4\cdot\frac{1}{4!}\left(\frac{1}{2}\right)^4$$

$\le \left(\frac{\pi}{2}\right)^4\cdot 1$ because

$$y^{(4)}(x) = \left(\frac{\pi}{2}\right)^4\cos\left(\frac{\pi}{2}x\right)$$

Thus $|\text{error}| \le \dfrac{\pi^4}{16^2\cdot 24} = \underline{0.0158543}$

Exact error at $x=\frac{1}{2}$ $\Big\} = \underbrace{\cos\frac{\pi}{4}}_{\sqrt{2}/2} - p\left(\frac{1}{2}\right) = \underline{0.0107572}$

((Compare with P2 from exercises 1))

8. (+) Avoiding Runge phenomenon

(−) Expense or impossibility of measurement. High density / concentration at boundaries of x-range.
Not embedded when the # of args is increased.

3 add: $|\text{Error}| \le \max_{x\in[0;1]}|y^{(6)}(x)|\ /\ |x^3(x-1)^3|\ /\cdot\frac{1}{6!}$

$$\le \frac{M}{6!}\cdot\underbrace{|x^3(x-1)^3|}_{\le\left(\frac{1}{2}\right)^6} \le 10\left(\frac{1}{2}\right)^6\cdot\frac{1}{6!}$$

$\underline{0.000217}$

(Sol. 2 cont.)

5 add

a) $|\text{Error}| \leq \underbrace{\overbrace{\max_{x\in[0;2]} |y^{(5)}(x)|}^{=M}}_{\leq 100} \cdot \underbrace{|x^2(x-1)(x-2)^2|}_{\text{max. at } x \overset{!}{\underset{@}{=}} \frac{1}{5}(5-\sqrt{5})} \cdot \frac{1}{5!}$

$\leq \underline{100 \cdot 0.286217 \cdot \frac{1}{120} = 0.238514}$

b) $|\text{Error}| \leq \underbrace{\max_{x\in[0;1]} |y^{(5)}(x)|}_{\leq M = 100} \cdot \underbrace{|x^3(x-1)^2|}_{\text{max at } x \underset{@}{=} \frac{3}{5}} \cdot \frac{1}{5!}$

$\leq \underline{100 \cdot 0.03456 \cdot \frac{1}{120} = 0.0288}$