Solutions 4

1. ▸ Data $n=3$; $h=\frac{\pi}{3}=$ const.; $x_0=0$; $x_1=\frac{\pi}{3}$; $x_2=\frac{2\pi}{3}$; $x_3=\pi$

▸ Splines $S_i(x) = a_i + b_i(x-x_i) + c_i(x-x_i)^2 + d_i(x-x_i)^3$ $(i=0,\dots,2)$

a) ▸ $a_i = y_i$ ✓ $(y_0=0;\ y_1=\frac{\sqrt{3}}{2};\ y_2=\frac{\sqrt{3}}{2};\ y_3=0)$

▸ $c_0=0$, linear system for $\{c_1,c_2\}$:

$$\begin{pmatrix} 2\frac{2\pi}{3} & \frac{\pi}{3} \\ \frac{\pi}{3} & 2\frac{2\pi}{3}\end{pmatrix}\begin{pmatrix} c_1 \\ c_2\end{pmatrix} = \begin{pmatrix} 3\left(\frac{a_2-a_1}{h} - \frac{a_1-a_0}{h}\right) \\ 3\left(\frac{a_3-a_2}{h} - \frac{a_2-a_1}{h}\right)\end{pmatrix} = \begin{pmatrix} -\frac{9\sqrt{3}}{2\pi} \\ -\frac{9\sqrt{3}}{2\pi}\end{pmatrix}$$

▸ $\Longrightarrow c_1 = \frac{-27\sqrt{3}}{\pi^2 10}$; $c_2 = \frac{-27\sqrt{3}}{10\pi^2}$ (© ...)

$\Longrightarrow d_0 = \frac{c_1-c_0}{3(\frac{\pi}{3})} = \frac{-27\sqrt{3}}{10\pi^3}$; $d_1 = \frac{c_2-c_1}{3\cdot h} = 0$ (1.2)

$\Longrightarrow d_2 = \frac{-c_2}{3h} = \frac{27\sqrt{3}}{\pi^3 10}$

$\Longrightarrow b_0 = \frac{a_1-a_0}{h} - \frac{2c_0+c_1}{3}\cdot h = \frac{9\sqrt{3}}{5\pi}$ (1.3)

$\Longrightarrow b_1 = \frac{a_2-a_1}{h} - \frac{2c_1+c_2}{3} h = \frac{9\sqrt{3}}{10\pi}$

$\Longrightarrow b_2 = \frac{a_3-a_2}{h} - c_2\frac{\pi}{3} - d_2\left(\frac{\pi}{3}\right)^2 = \frac{-9\sqrt{3}}{10\pi}$ (1.3')

▸ $S_0(x) = 0 + \frac{9\sqrt{3}}{5\pi}(x-0) + 0(x-0)^2 - \frac{27\sqrt{3}}{10\pi^3}(x-0)^3$ $\left(0 \le x \le \frac{\pi}{3}\right)$

$S_1(x) = \frac{\sqrt{3}}{2} + \frac{9\sqrt{3}}{10\pi}\left(x-\frac{\pi}{3}\right) - \frac{27\sqrt{3}}{10\pi^2}\left(x-\frac{\pi}{3}\right)^2 + 0\left(x-\frac{\pi}{3}\right)^3$ $\left(\frac{\pi}{3} \le x \le \frac{2\pi}{3}\right)$

$S_2(x) = \frac{\sqrt{3}}{2} - \frac{9\sqrt{3}}{10\pi}\left(x-\frac{2\pi}{3}\right) - \frac{27\sqrt{3}}{10\pi^2}\left(x-\frac{2\pi}{3}\right)^2 + \frac{27\sqrt{3}}{10\pi^3}\left(x-\frac{2\pi}{3}\right)^3$ $\left(\frac{2\pi}{3} \le x \le \pi\right)$

(Sol. 4 cont.)

1b) ▷ "Clamped cubic spline" $S_0'(0) = y_0' = 1$
$S_2'(\pi) = y_3' = -1$

▷ Incomplete linear system for c_0, c_1, c_2:

$$\frac{\pi}{3} c_0 + 2\left(\frac{\pi}{3}+\frac{\pi}{3}\right)c_1 + \frac{\pi}{3}c_2 = 3\left(\frac{y_2-y_1}{h} - \frac{y_1-y_0}{h}\right) \iff$$

$$\iff \frac{\pi}{3}c_0 + \frac{4\pi}{3}c_1 + \frac{\pi}{3}c_2 = 3\left(\frac{3}{\pi}\cdot 0 - \frac{3}{\pi}\frac{\sqrt{3}}{2}\right) = \frac{-9\sqrt{3}}{2\pi} \quad \text{(I)}$$

▷ (1.6) $$2c_0\frac{\pi}{3} + c_1\frac{\pi}{3} = 3\left(\frac{y_1-y_0}{h} - y_0'\right) = \frac{9\sqrt{3}}{2\pi} - 3 \quad \text{(II)}$$

▷ (1.7) $$2c_1\frac{\pi}{3} + 4c_2\frac{\pi}{3} + 3c_2\frac{\pi}{3} = 9\left(\frac{y_3-y_2}{h}\right) - 6\left(\frac{y_2-y_1}{h}\right) - 3y_3'$$

$$\iff 2c_1\frac{\pi}{3} + c_2\frac{7\pi}{3} = 9\left(\frac{-\sqrt{3}\cdot 3}{2\pi}\right) - 6\cdot 0 + 3 = \frac{-27\sqrt{3}}{2\pi} + 3 \quad \text{(III)}$$

$$\text{I, II, III} \Longrightarrow c_0, c_1, c_2 \xrightarrow{(1.2)} d_0, d_1 \xrightarrow{(1.3)} b_0, b_1 \xrightarrow{(1.4)} b_2 \xrightarrow{(1.3')} d_2$$

Numerically (c)

$$c_0 = \frac{-10\pi + 18\sqrt{3}}{2\pi^2}; \quad c_1 = \frac{2\pi - 9\sqrt{3}}{2\pi^2}; \quad c_2 = \frac{2\pi - 9\sqrt{3}}{2\pi^2}$$

$$d_0 = \frac{c_1 - c_0}{3h} = \frac{1}{\pi}\,\frac{12\pi - 27\sqrt{3}}{2\pi^2}; \quad d_1 = \frac{c_2 - c_1}{3h} = \frac{1}{\pi}\cdot 0 = 0$$

$$b_0 = \frac{y_1 - y_0}{h} - \frac{2c_0 + c_1}{3}h = \frac{3}{\pi}\,\frac{\sqrt{3}}{2} - \frac{-18\pi + 27\sqrt{3}}{2\pi^2\cdot 3}\cdot\frac{\pi}{3} =$$

$$= \frac{6\pi - 0\sqrt{3}}{2\pi\cdot 3} = 1$$

$$b_1 = \frac{y_2 - y_1}{h} - \frac{2c_1 + c_2}{3}h = 0 - \frac{6\pi - 27\sqrt{3}}{2\pi^2\cdot 3}\cdot\frac{\pi}{3} = -\frac{2\pi - 9\sqrt{3}}{6\pi}$$

$$b_2 = b_1 + 2c_1 h + 3d_1 h^2 = \ldots = \frac{1}{3} - \frac{3\sqrt{3}}{2\pi} \quad (1.4)$$

d_2: $$b_2 = \frac{y_3 - y_2}{h} - c_2 h - d_2 h^2 \Longrightarrow$$

$$d_2 = \left(\frac{y_3 - y_2}{h} - c_2 h - b_2\right)/h^2 = \ldots = \frac{27\sqrt{3} - 12\pi}{2\pi^3}$$

(Sol. 4 cont.)

1 c) ⚠ y is a C^4 function, and (!):

- S coincides with $y'' = 0$ at the boundaries in the case of problem a).
- S coincides with $y' \in \{1, -1\}$ at the boundaries in case b).

So by theory:

- $|y - S| \leq \max |y^{(4)}(x)| \cdot \frac{5}{384} H^4 \qquad (H = \max h_i)$

$$= \max |\sin(x)| \cdot \frac{5}{384} \cdot \left(\frac{\pi}{3}\right)^4$$
$$= 1 \cdot \frac{5}{384} \cdot \frac{\pi^4}{81} \approx \underline{0.015659}$$

- $|y' - S'| \leq \max |y^{(4)}(x)| \frac{H^3}{24} = 1 \cdot \frac{\pi^3}{27 \cdot 24} \approx \underline{0.047849}$

- $|y'' - S''| \leq \max |y^{(4)}(x)| \frac{3}{8} H^2 = 1 \cdot \frac{3}{8} \frac{\pi^2}{9} \approx \underline{0.411234}$

---

2 $n = 8$; $i = 0, \ldots, n-1 = 7$ = # spline patches.
Linear system for $\{c_1, \ldots, c_7\}$; $(c_0 = 0)$:

$$\begin{pmatrix} 2(h_0+h_1) & h_1 & 0 & 0 & 0 & 0 & 0 \\ h_1 & 2(h_1+h_2) & h_2 & 0 & 0 & 0 & 0 \\ 0 & h_2 & 2(h_2+h_3) & h_3 & 0 & 0 & 0 \\ 0 & 0 & h_3 & 2(h_3+h_4) & h_4 & 0 & 0 \\ 0 & 0 & 0 & h_4 & \boxed{\ldots} & h_5 & 0 \\ 0 & 0 & 0 & 0 & h_5 & \boxed{\ldots} & h_6 \\ 0 & 0 & 0 & 0 & 0 & h_6 & \underset{\uparrow\ 2(h_6+h_7)}{\boxed{\ldots}} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \\ c_7 \end{pmatrix} = \ldots$$

$h_0 = \Delta x_0 = 2$; $h_1 = \Delta x_1 = 0.5$ a.s.o.

(Sol. 4 cont.)

2 (cont.)

$$\ldots = \begin{pmatrix} 3\left(\frac{0.6}{0.5} - \frac{2.9}{2}\right) \\ \vdots \\ \vdots \\ 3\left(\frac{-2.6}{1} - \frac{-0.9}{0.5}\right) \end{pmatrix}$$

The r.h.s. is computed by Table 1.3 :

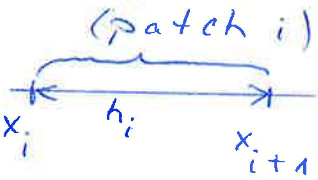
$$3\left(\frac{y_{i+1} - y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right) \qquad (i = 1, \ldots, n-1 = 7).$$

3

$$S_i' = b_i (x - x_i)^0 + 2c_i (x - x_i) + 3d_i (x - x_i)^2$$

$$S_i'' = 2c_i + 6d_i (x - x_i) \; ; \; S_i''' = 6d_i$$

a) Substitute $x = x_i$:



$$S_i(x_i) = a_i$$

$$S_i'(x_i) = b_i \; ; \; S''(x_i) = 2c_i \; ; \; S'''(x_i) = 6d_i$$

b) Substitute $x = x_{i+1}$:

$$S_i(x_{i+1}) = a_i + b_i h_i + c_i h_i^2 + d_i h_i^3$$

$$S_i'(\ldots) = b_i + 2c_i h_i + 3d_i h_i^2$$

$$S_i''(\ldots) = 2c_i + 6d_i h_i^1$$

$$S_i'''(\ldots) = 6d_i$$

---

4 a) (1) Logics: Since a straight line has its second derivatives equal to 0 it represents the natural spline.

Thus $S(x) = y(x) = x + 1$, more precisely:

$$S_0 = x + 1 \quad (-1 \le x \le 0)$$

$$S_1 = x + 1 \quad (0 \le x \le 1)$$

$$S_2 = x + 1 \quad (1 \le x \le 3)$$

(Sol. 4 cont.)

4 (cond.) c)a) (2) Algebra The right hand side of the linear system contains the values $3\left(\frac{y_{i+1}-y_i}{h_i} - \frac{y_i - y_{i-1}}{h_{i-1}}\right)$ $(i=1,\dots,n-1)$. They all are 0 because $\frac{\Delta y_i}{\Delta h_i}$ is the constant slope of the straight line $(i=0,\dots,n-1)$.

b) No computations required! $y'' = \text{const.} \neq 0$ for any parabola of order 2, thus no natural spline.

c) (rev.) Since $c_i = 0$ $(i=0,\dots,n-1)$ (1.2) implies $d_i = 0$ $(i=0,\dots,n-2)$ and $d_{n-1} = \frac{-c_{n-1}}{3h_{n-1}} = 0$ (natural spline).

5* Cf. the hint.

6* " " "