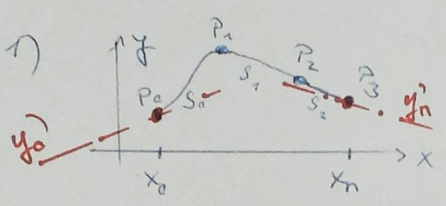
Solutions 4 Extended (Problems 5, 6)

1 Numerical Analysis & Comp. Alg. BZ

Solutions 4 (extended)

5.** The statement has physical consequences:

(1)



The function $y = S(x)$ of a highly elastic rod through fixed points $P_0; P_1; P_2; \ldots$ is a natural spline.

(2) The function $y = f(x)$ of a highly elastic rod through fixed pts $P_0; P_1; P_2; \ldots$ and given slopes y_0' and y_n' at P_0 and P_n, respectively, is a complete ("clamped") cubic spline.
Here again, the mean total curvature $\sim \int_{x_0}^{x_n} |f''(x)|^2 dx$ is minimized by the spline function.

6.** (Sol.) $\int_{x_0}^{x_n} f'(x)^2 - \int_{x_0}^{x_n} S'(x)^2 \overset{!}{=} \int (f'-S')^2 + 2\underbrace{\int S'(f'-S')}_{R}$

$R = 0$ because: $\int_{x_0}^{x_n} S'(f'-S') = S'(f-S)\Big|_{x_0}^{x_n} -$

$- \int_{x_0}^{x_n} S''(f-S) = 0 - 0$ as $f(x_0) = S(x_0)$ &

$f(x_n) = S(x_n)$ and $S'' = 0^{*)}$ (S is piecewise linear).

Then $\int f'^2 - \int S'^2 = \underbrace{\int (f'-S')^2}_{\geq 0} \geq 0 \Rightarrow \int_{x_0}^{x_n} f'^2 \geq \int_{x_0}^{x_n} S'^2$ ☒

[*) $\int_{x_0}^{x_n} S''(f-S) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \overset{=0}{S''}(f-S) = 0$]

5** From Schaum's Outline of Numerical Analysis 2nd ed. (Problem 9.17)

First note that

$$\int_a^b f''(x)^2\,dx - \int_a^b S''(x)^2\,dx = \int_a^b [f''(x) - S''(x)]^2\,dx + 2\int_a^b S''(x)[f''(x) - S''(x)]\,dx$$

with $S(x)$ the cubic spline. Integration by parts over each subinterval converts the last integral as follows:

$$\int_{x_{i-1}}^{x_i} S_i''(x)[f''(x) - S_i''(x)]\,dx = S_i''(x)[f'(x) - S_i'(x)]\,\Big|_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} [f'(x) - S_i'(x)]S_i^{(3)}(x)\,dx$$

$$= S_i''(x)[f'(x) - S_i'(x)]\,\Big|_{x_{i-1}}^{x_i} - S_i^{(3)}(x)[f(x) - S_i(x)]\Big|_{x_{i-1}}^{x_i} + \int_{x_{i-1}}^{x_i} [f(x) - S_i(x)]S_i^{(4)}(x)\,dx$$

The last two terms vanish since $f(x)$ equals $S_i(x)$ at the knots and $S_i^{(4)}(x)$ is zero. Summing what is left for $i = 1, \ldots, n$ there is cancellation of all interior values leaving

$$S''(b)[f'(b) - S'(b)] - S''(a)[f'(a) - S'(a)]$$

which also vanishes since S is the natural spline. Notice that this remnant would still vanish if we assumed instead that f' and S' agree at the endpoints. In either case, reordering the original equation just slightly,

$$\int_a^b S''(x)^2\,dx = \int_a^b f''(x)^2\,dx - \int_a^b [f''(x) - S''(x)]^2\,dx$$

which does make the first integral smaller than the second.