Solutions 5

1 a) · The spline S_0 from problem 4.1 is

$$S_0(x) = \frac{9\sqrt{3}\,x}{5\pi} - \frac{27\sqrt{3}}{10\pi^3}x^3 \qquad \left(0 \le x \le \frac{\pi}{3}\right)$$

· The spline from Example 2.3 of the spline interpolation script is represented in vector parameter form:

$$\vec{r}_1(t) = \begin{pmatrix} \frac{\pi}{9} B_{1,3}(t) + \frac{2\pi}{9} B_{2,3}(t) + \frac{1}{3}\pi B_{3,3}(t) \\ \frac{\sqrt{3}}{5} B_{1,3}(t) + \frac{2}{5}\sqrt{3} B_{2,3}(t) + \frac{\sqrt{3}}{2} B_{3,3}(t) \end{pmatrix} \qquad (0 \le t \le 1)$$

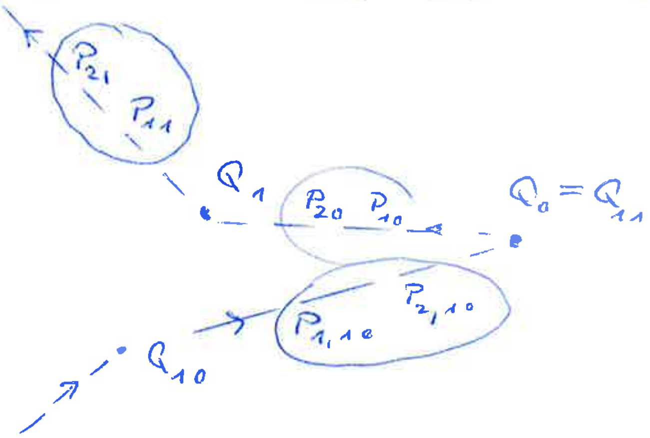
$$= \begin{pmatrix} \frac{\pi}{9} 3(1-t)^2 t + \frac{2\pi}{9} 3(1-t)t^2 + \frac{1}{3}\pi t^3 \\ \frac{\sqrt{3}}{5} 3(1-t)^2 t + \frac{2}{5}\sqrt{3}\, 3(1-t)t^2 + \frac{\sqrt{3}}{2} t^3 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\overset{!}{\Longleftrightarrow} \begin{pmatrix} \frac{\pi}{3} t = x \\ \frac{3\sqrt{3}}{5} t - \frac{\sqrt{3}}{10} t^3 = y \end{pmatrix} \Longrightarrow \begin{pmatrix} t = \frac{3x}{\pi} \\ y = \frac{9\sqrt{3}}{5\pi} x - \frac{27\sqrt{3}}{10\pi^3} x^3 \end{pmatrix} \checkmark$$

b) Because of the coincidence in a) we can conclude from Ex. 4.1c) that

$$|y - S| \le \frac{5}{384} \cdot \frac{\pi^4}{81} \approx 0.015659 \qquad (0 \le x \le \pi)$$

2



a) Equations

(Q_1) C^1: $P_{11} - Q_1 = Q_1 - P_{20}$

C^2: $P_{21} - 2P_{11} + Q_1 = Q_1 - 2P_{20} + P_{10}$

(Q_0) Cusp (!)

$P_{10} - Q_0 = \vec{0}$ (!)

$Q_0 - P_{2,10} = \vec{0}$ (!)

b) $22 = 11 \cdot 2$ interior control points because there are 11 "patches".

c) 22 (!) : $10 \cdot 2 = 20$ equations for $Q_1, \ldots, Q_{10}$

(Sol. 5 cont.)

2 (cont.) c) (cont.) ... 2 equations at Q_0.

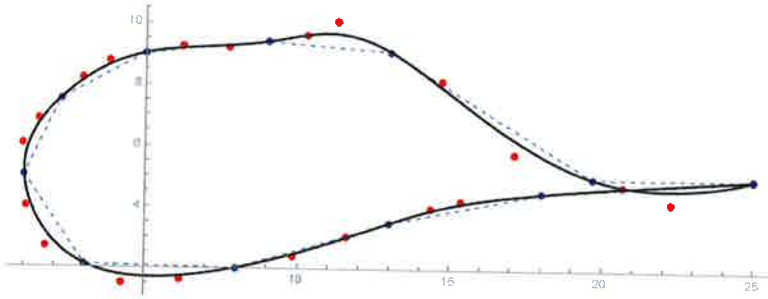
d) Q_0: $Q_0 - P_{2,10} = 0$, $P_{10} - Q_e = 0$

Q_j: $P_{2_j} - 2P_{1_j} + Q_j = Q_j - 2P_{2,j-1} + P_{1,j-1}$
$(j = 1, \dots, 10)$

System of 22 linear equations in the vars $\{P_{1_j}, P_{2_j} \mid j = 0, \dots, 10\}$.

Cf. Exercises5P2.nb for an implementation in Mathematica.
The two spline patches involving Q_0 intersect (beside Q_0).

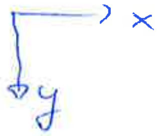
The figure below represents all splines and control points. Note the singular situation at the cusp (Q_e).



(Sol. 5 cont.)

3 Three splines (c means cubic Bézier splines with relative = incremental coordinates).

(1) $A = M = \begin{pmatrix}183.171\\159.905\end{pmatrix} = P_0$



$P_1 = A + \begin{pmatrix}-19.081\\16.712\end{pmatrix}$; $P_2 = A + \begin{pmatrix}-71.022\\29.131\end{pmatrix}$

$P_3 = A + \begin{pmatrix}-71.022\\80.636\end{pmatrix}$

$\vec{r}_1(t) = P_0 B_{03}(t) + P_1 B_{13}(t) + P_2 B_{23}(t) + P_3 B_{33}(t)$

$(0 \le t \le 1)$

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(2) $A = Q_0 = P_3$ from (1) above.

$Q_1 = A + \begin{pmatrix}0\\22.975\end{pmatrix}$; $Q_2 = A + \begin{pmatrix}11.665\\34.132\end{pmatrix}$; $Q_3 = A + \begin{pmatrix}27.556\\34.132\end{pmatrix}$

$\vec{r}_2(t) = Q_0 B_{03}(t) + Q_1 B_{13}(t) + Q_2 B_{23}(t) + Q_3 B_{33}(t)$

$(0 \le t \le 1)$

(3) a.s.o. $A = R_0 = Q_3$ (from (2) above)

Cf. Case_FontDesignBezier1.nb for an implementation in Mathematica.

Smoothness checking (C^0 is fulfilled, of course)

(C^1): $\vec{r}_1{}'(1) = \begin{pmatrix}0\\154.515\end{pmatrix}$; $\vec{r}_2{}'(0) = \begin{pmatrix}0\\68.925\end{pmatrix}$ (failed)

$\vec{r}_2{}'(1)$ $\begin{pmatrix}47.673\\0\end{pmatrix}$; $\vec{r}_3{}'(0) = \begin{pmatrix}46.116\\0\end{pmatrix}$ (failed)

C^1 fails but the derivatives are collinear in either case. Thus the splines have a common tangent line (no "crinkle").

(C^2) $\vec{r}_1{}''(1) = \begin{pmatrix}311.6\\234.5\end{pmatrix}$; $\vec{r}_2{}''(0) = \begin{pmatrix}64.99\\-70.91\end{pmatrix}$ (failed)

$\vec{r}_2{}''(1) = \begin{pmatrix}25.4\\-66.9\end{pmatrix}$; $\vec{r}_3{}''(0) = \begin{pmatrix}76.4\\-130.3\end{pmatrix}$ (") □

(Sol. 5 cont.)

3 (cont.) The figure below shows the plots of $\vec{r}_1, \vec{r}_2, \vec{r}_3$ with reflected y coordinate.

Note the common tangent lines at the interior contact points.