Solutions 6

1 $N=4$, $m=2$. Basis $\{g_0, g_1, g_2\} = \{1, x, x^2\}$

a) Design matrix G

$$\begin{array}{c|ccc} & g_0 & g_1 & g_2 \\ \hline x_0 & 1 & 2 & 4 \\ x_1 & 1 & 3 & 9 \\ x_2 & 1 & 4 & 16 \\ x_3 & 1 & 5 & 25 \\ x_4 & 1 & 6 & 36 \end{array}$$

Normal equations: $G^t G\, a = G^t y$

$$\begin{pmatrix} 5 & 20 & 90 \\ 20 & 90 & 440 \\ 90 & 440 & 2274 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 9.75 \\ 41.66 \\ 196.4 \end{pmatrix}$$

b) $a_0 = 0.506$; $a_1 = 0.483143$; $a_2 = -0.0271429$

$y = a_0 \cdot 1 + a_1 \cdot x + a_2 x^2$

c) $y(4) = 2.00429$; $y'(4) \overset{!}{=} a_1 + 2a_2 \cdot 4 = 0.266$

2 Data $\{x_0, x_1, x_2, \ldots, x_N\}$; $\Delta x = h$

Windows consisting of 5 consecutive points:

$\{x_0, x_1, x_2, x_3, x_4\}$ $(k=2)$ Central coord. x_2

$\{x_1, x_2, x_3, x_4, x_5\}$ $(k=3)$ " " x_3

$\{x_2, x_3, x_4, x_5, x_6\}$ $(k=4)$ center x_4 . . . $\{x_{N-4}, \ldots, x_N\}$ $k = N-2$; center x_{N-2}

A set $\{x_{k-2}, x_{k-1}, x_k, x_{k+1}, x_{k+2}\}$ of 5 points is transformed to integer coordinates $\{-2,-1,0,1,2\}$ by $X = \dfrac{x - x_k}{h}$. The least-squares approximation is computed in the variable X: $y = a_0 + a_1 X + a_2 X^2$

$y(0) = a_0$; $y'(0) \overset{!}{=} a_1$; $(y''(0) = 2a_2)$.

(Sol. 6 cont.)

2 (cont.) a) $G = \begin{pmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ Design matrix

$G^t G\, a| = G^t y|$ Normal equations

$$\begin{aligned} &\text{I} && 5a_0 + 0a_1 + 10a_2 = y_{k-2} + y_{k-1} + y_k + y_{k+1} + y_{k+2} \\ &\text{II} && 0a_0 + 10a_1 + 0a_2 = -2y_{k-2} - y_{k-1} + y_{k+1} + 2y_{k+2} \\ &\text{III} && 10a_0 + 0a_1 + 34a_2 = 4y_{k-2} + y_{k-1} + y_{k+1} + 4y_{k+2} \end{aligned}$$

b) $\text{II} \Rightarrow a_1 = \frac{1}{10}(-2y_{k-2} - y_{k-1} + y_{k+1} + 2y_{k+2})$

$-2\,\text{I} + \text{III} \Rightarrow \dots$

$\dots\ a_2 = \frac{1}{14}(2y_{k-2} - y_{k-1} - 2y_k - y_{k+1} + 2y_{k+2})$

$-5\,\text{III} + 17\,\text{I} \Rightarrow \dots$

$$\begin{aligned} \dots\ a_0 &= \frac{1}{35}(-3y_{k-2} + 12y_{k-1} + 17y_k + 12y_{k+1} - 3y_{k+2}) \\ &\overset{!!}{=} y_k - \frac{3}{35}(y_{k-2} - 4y_{k-1} + 6y_k - 4y_{k+1} + y_{k+2}) \\ &= y_k - \frac{3}{35}\Delta^4 y_{k-2} = y_k - \frac{3}{35}\delta^4 y_k \end{aligned}$$

c) Therefore: $(k = 2, \dots, N-2)$

$$\begin{aligned} y &= y(x_k) = y(\underset{t=0}{0}) \approx y_k - \frac{3}{35}\Delta^4 y_{k-2} \\ y' &= y'(x_k) \overset{!}{=} \frac{1}{h} y'(0) \approx \frac{1}{h} a_1 = \dots \\ &\dots = \frac{1}{10h}(-2y_{k-2} - y_{k-1} + y_{k+1} + 2y_{k+2}) \\ y'' &= y''(x_k) = \frac{1}{h^2} y''(0) \approx \frac{1}{h^2} 2 a_2 = \dots \\ &\dots = \frac{2}{14h^2}(2y_{k-2} - y_{k-1} - 2y_k - y_{k-1} + 2y_{k+2}) \end{aligned}$$

Note that the (inner) derivative of the transformation $t = \frac{x - x_k}{h}$ with respect to x is equal to $\frac{1}{h}$.
The exercise requires the derivatives with respect to x, finally.

(Sol. 6 cont.)

2 (cont.) d) $y(x_{k-2}) \hat{=} y(t=-2) \approx a_0 + a_1 t + a_2 t^2 \big|_{t=-2}$

$= a_0 - 2a_1 + 4a_2$

$y'(x_{k-2}) \overset{!}{=} \underbrace{\frac{1}{h}(a_1 + 2a_2 t)\Big|_{t=-2}}_{y'(t=-2)} = \frac{1}{h}(a_1 - 4a_2)$

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3 By 2 we know that $y(x_k) \approx a_0 = y_k - \frac{3}{35}\Delta^4 y_{k-2}$ for $k = 2, \dots, N-2 = 7$.

Here: $\{x_0, x_1, x_2, \dots, x_{N=9}\} = \{1, 2, 3, \dots, 10\}$

Computation is even possible by hand (!)

x_i	1	2	3 (x_2)	4	5	6	7	8 (x_7)	9	10
y_i	1.04	1.37	1.70	2	2.26	2.42	2.7	2.78	3	3.14
Δy_i	0.33	0.33	0.3	0.26	0.16	0.28	0.08	0.22	0.14	
$\Delta^2 y_i$	0	-0.03	-0.04	-0.1	0.12	-0.2	0.14	-0.08		
$\Delta^3 y_i$	-0.03	-0.01	-0.06	0.22	-0.32	0.34	-0.22			
$\Delta^4 y_i$	0.02	-0.05	0.28	-0.54	0.66	-0.56				

Now:

$k=2 \Rightarrow x=3: \quad a_0 = y_2 - \frac{3}{35}\Delta^4 y_0 = 1.70 - \frac{3}{35}\,0.02 = 1.6983$

$k=3 \Rightarrow x=4: \quad a_0 = y_3 - \frac{3}{35}\Delta^4 y_1 = 2 - \frac{3}{35}(-0.05) = 2.0043$

$k=4 \Rightarrow x=5: \quad a_0 = y_4 - \frac{3}{35}\Delta^4 y_2 = 2.26 - \frac{3}{35}(0.28) = 2.236$

$k=5 \Rightarrow x=6: \quad a_0 = y_5 - \frac{3}{35}\Delta^4 y_3 = 2.42 - \frac{3}{35}(-0.54) = 2.4663$

$k=6 \Rightarrow x=7: \quad a_0 = y_6 - \frac{3}{35}\Delta^4 y_4 = 2.7 - \frac{3}{35}(0.66) = 2.6434$

$k=7 \Rightarrow x=8: \quad a_0 = y_7 - \frac{3}{35}\Delta^4 y_5 = 2.78 - \frac{3}{35}(-0.56) = 2.828$

(Sol. 6 cont.)

4 Data window of 5 points: $\{x_0, \dots, x_4\} = \{2, \dots, 6\}$.
$N = 4$; transformation $t = \frac{x-2}{1} = x - 2 \in \{0, \dots, 4\}$.

Orthogonal polys:

$P_{0,4}(t) = 1$; $P_{1,4}(t) = 1 - \frac{t}{2}$; $P_{2,4}(t) = 1 - \frac{3}{2}t + \frac{1}{2}t(t-1)$

Design matrix: $G = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1/2 & -1/2 \\ 1 & 0 & -1 \\ 1 & -1/2 & -1/2 \\ 1 & -1 & 1 \end{pmatrix}$

Normal matrix: $G^t G = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5/2 & 0 \\ 0 & 0 & 7/2 \end{pmatrix}$ (!)

R.h.s.: $G^t y| = \begin{pmatrix} 9.75 \\ -1.33 \\ -0.19 \end{pmatrix}$

$\Longrightarrow 5a_0 = 9.75 \Longrightarrow a_0 = 1.95$

$\frac{5}{2}a_1 = -1.33 \Longrightarrow a_1 = -2.66/5 = -0.532$

$\frac{7}{2}a_2 = -0.19 \Longrightarrow a_2 = -\frac{0.38}{7} = -0.0\cancel{76}54$

$y(t) = a_0 \cdot 1 + a_1 P_{1,4}(t) + a_2 P_{2,4}(t)$

$y(x=4) \overset{!}{=} y(t=2) = a_0 + a_1 \cdot 0 + a_2(-1) = 2.00429$ ✓

$y'(x=4) = \left(\frac{1}{1}\right) y'(t=2) = a_1 P_{1,4}'(2) + a_2 P_{2,4}'(2)$

$= a_1\left(-\frac{1}{2}\right) + \left(\cancel{-\frac{3}{2} + 2\frac{1}{2}}\right) = 0.266$ ✓

$-2+2=0$

(Sol. 6 cont.)

5 a) G

$$
\begin{array}{c|ccc}
x & 1 & -\frac{x}{2} & \frac{x^2}{2}-1 \\
\hline
-2 & 1 & 1 & 1 \\
-1 & 1 & \frac{1}{2} & -\frac{1}{2} \\
0 & 1 & 0 & -1 \\
1 & 1 & -\frac{1}{2} & -\frac{1}{2} \\
2 & 1 & -1 & 1
\end{array}
$$

$G^t G$

$$
\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5/2 & 0 \\ 0 & 0 & 7/2 \end{pmatrix}
$$

Diagonal!

b) $y = a_0 \cdot 1 + a_1 \left(-\frac{x}{2}\right) + a_2 \left(\frac{x^2}{2} - 1\right)$

R.h.s. of normal equations $= G^t y / =$

$$
\begin{pmatrix} 0 + 1 + 2 + 3 + 1 \\ 1\cdot 0 + \frac{1}{2}\cdot 1 + 0\cdot 2 - \frac{1}{2}\, 3 - 1\cdot 1 \\ 1\cdot 0 - \frac{1}{2}(+1) - 1\cdot 2 - \frac{1}{2}\cdot 3 + 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ -\cancel{2}3 \end{pmatrix}
$$

$a_0 = \frac{7}{5}$; $a_1 = -2/(5/2) = -\frac{4}{5}$; $a_2 = -\cancel{2}3/(7/2) = -\frac{\cancel{4}6}{7}$

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c) $G = U D V^t$; $U : 5\times 5$; $D = 5\times 3$; $V = 3\times 3$

d) The singular values are the roots of the eigenvalues of $G^t G$, thus $\{\sqrt{5} ; \sqrt{5/2} ; \sqrt{7/2}\}$.

$$
D = \begin{pmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{5/2} & 0 \\ 0 & 0 & \sqrt{7/2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

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e) $\{1 ; -\frac{x}{2} ; \frac{x^2}{2} - 1\}$ is orthogonal because $G^t G$ is diagonal (cf. a))