Solutions 7

1 $T_0 = \cos(0 \cdot \arccos x) = \cos 0 = 1$ ✓

$T_1 = \cos(1 \cdot \arccos x) \stackrel{!}{=} x$ ✓

$T_2 = \cos 2a = 2\cos^2 a - 1 = 2x^2 - 1$ ✓ if $a = \arccos x$

$T_3 = \cos 3a = 4\cos^3 a - 3\cos a = 4x^3 - 3x$ if ''

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2 $n = 3 \Rightarrow T_{4=3+1} = 2x \overset{\checkmark}{T_3} - \overset{\checkmark}{T_2} = 2x(4x^3 - 3x) - (2x^2 - 1)$
$= \underline{8x^4 - 8x^2 + 1}$

$n = 4 \Rightarrow T_5 = 2x\,T_4 - T_3 = 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x)$
$= \underline{16x^5 - 20x^3 + 5x}$

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3 $1 = T_0$; $x = T_1$; $T_2 = 2x^2 - 1 \Rightarrow x^2 = \dfrac{T_2 + 1}{2}$, $1 = T_0$

$T_3 = 4x^3 - 3x \Rightarrow x^3 = \dfrac{T_3 + 3x}{4} = \dfrac{T_3 + 3T_1}{4}$

$T_4 = 8x^4 - 8x^2 + 1 \Rightarrow x^4 = \dfrac{(T_4 + 8x^2 - 1)}{8} = \dfrac{(T_4 + 4(T_2 + T_0) - T_0)}{8}$

$= \dfrac{T_4 + 4T_2 + 3T_0}{8}$

$T_5 = 16x^5 - 20x^3 + 5x$

$\Rightarrow x^5 = \dfrac{(T_5 + 20x^3 - 5x)}{16} = \dfrac{(T_5 + 5T_3 + 15T_1 - 5T_1)}{16}$

$= \dfrac{T_5 + 5T_3 + 10T_1}{16}$

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4 $P_0 = \dfrac{1}{1 \cdot 1} \cdot 1 = 1$ ✓ ; $P_1 = \dfrac{1}{2 \cdot 1} \dfrac{d}{dx}(x^2 - 1) = \frac{1}{2} \cdot 2x = x$ ✓

$P_2 = \dfrac{1}{4 \cdot 2} \cdot \dfrac{d^2}{dx^2}(x^4 - 2x^2 + 1) = \frac{1}{8}(4 \cdot 3x^2 - 4) = \frac{3}{2}x^2 - \frac{1}{2}$ ✓

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5 $P_3 \Rightarrow n = 2 : 3P_3 = 5x P_2 - 2P_1 = 5x\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - 2x$

$P_3 = \underline{\frac{5}{2}x^3 - \frac{3}{2}x}$ ✓

$P_4 \Rightarrow n = 3 : 4P_4 = 7x P_3 - 3P_2 = 7x\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) - 3\left(\frac{3}{2}x^2 - \frac{1}{2}\right)$

$P_4 = \underline{\frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8}}$ ✓

(Sol. 7 cont.)

6 $1 = P_0$; $x = P_1$; $P_2 = \frac{3}{2}x^2 - \frac{1}{2} \Rightarrow x^2 = \frac{2P_2 + (1=P_0)}{3}$

$P_3 = \frac{5}{2}x^3 - \frac{3}{2}x \Rightarrow x^3 = \frac{2P_3 + 3P_1}{5}$

$P_4 = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8} \Rightarrow x^4 = \frac{8P_4 + 30x^2 - 3}{35} =$

$x^4 = \frac{8P_4 + 20P_2 + 7P_0}{35}$

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7 $y = x^3$ ($-1 \le x \le 1$ standard range !)

a) $x^3 \stackrel{!}{=} \frac{T_3 + 3T_1}{4} = \frac{3}{4}x + \frac{1}{4}T_3$

Line $= \frac{3}{4}x$; $|\text{error}| \stackrel{!}{=} |\frac{1}{4}T_3| \le \frac{1}{4} \to$ c)

b) $x^3 \stackrel{!}{=} \frac{2P_3 + 3P_1}{5} = \frac{3}{5}x + \frac{2}{5}P_3$

Line $= \frac{3}{5}x$; $|\text{error}| \stackrel{!}{=} |\frac{2}{5}P_3| \le \frac{2}{5} \to$ c)

c) Already done in ab)

d) Plots cf. separated document.

Chebyshev: Better at boundary
More uniform overall ("equal ripple")
Worse in the mid range

Legendre: Worse at boundary
Not uniform overall
Better in the mid range

(Sol. 7 cont.)

8 $y(\mathcal{x}) = \mathcal{x}^2 \quad (0 \le \mathcal{x} \le 1)$ Non-standard range.

$\mathcal{x} = \frac{1}{2}(x+1) \quad (-1 \le x \le 1$: standard range$)$

a) $y = \frac{1}{4}x^2 + \frac{1}{2}x + \frac{1}{4} \stackrel{!}{=} \frac{1}{4}\,\frac{1}{2}(T_2 + T_0) + \frac{1}{2}T_1 + \frac{1}{4}T_0 =$

$= \frac{3}{8}T_0 + \frac{1}{2}x + \frac{1}{8}T_2 \Rightarrow$ line $= \frac{3}{8} + \frac{1}{2}x$

$|\text{Error}| \stackrel{!}{=} \left|\frac{1}{8}T_2\right| \le \frac{1}{8} \to c)$

line $= \frac{3}{8} + \frac{1}{2}(2\mathcal{x}-1)$

b) $y \stackrel{!}{=} \frac{1}{4}\left(\frac{2P_2+1}{3}\right) + \frac{1}{2}P_1 + \frac{1}{4}P_0 = \frac{1}{3}P_0 + \frac{1}{2}P_1 + \frac{1}{6}P_2 \Rightarrow$

$\Rightarrow$ line $= \frac{1}{3} + \frac{1}{2}x = \frac{1}{3} + \frac{1}{2}(2\mathcal{x}-1)$

$|\text{error}| \stackrel{!}{=} \left|\frac{1}{6}P_2\right| \le \frac{1}{6} \to c)$

c) Already done in ab) d) As in 7d)

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9 $y = \sin\mathcal{x} \quad (0 \le \mathcal{x} \le \pi)$ Non-standard range

$\mathcal{x} = \frac{\pi}{2}(x+1) \iff x = \frac{2}{\pi}\mathcal{x} - 1 \quad (x \in (-1,1))$

a) $P_3 = a_0 + a_1 T_1 + a_2 T_2 + a_3 T_3$

$a_0 \stackrel{!}{=} \frac{1}{\pi}\int_{-1}^{1} \sin\left(\frac{\pi}{2}(x+1)\right)\frac{1\,dx}{\sqrt{1-x^2}} \stackrel{!}{=} J(0,\tfrac{\pi}{2}) \approx 0.472001$

$a_1 = \frac{2}{\pi}\int_{-1}^{1} \sin(\ldots)\,x\,\frac{1\,dx}{\sqrt{1-x^2}} \stackrel{!}{=} 0$

$a_2 = \frac{2}{\pi}\int_{-1}^{1} \sin(\ldots)\,T_2\,\frac{1\,dx}{\sqrt{1-x^2}} \stackrel{!}{=} \frac{2}{\pi}\left(\int_{-1}^{1}\sin(\ldots)\frac{2x^2\,dx}{\sqrt{1-x^2}} - \int_{-1}^{1}\sin(\ldots)\frac{1\,dx}{\sqrt{1-x^2}}\right)$

$\stackrel{!}{=} \frac{2}{\pi}\left(2\cdot 2J(1,\tfrac{\pi}{2}) - 2\pi J(2,\tfrac{\pi}{2}) - \pi J(0,\tfrac{\pi}{2})\right) \approx -0.499403$

$a_3 = \frac{2}{\pi}\int_{-1}^{1} \sin(\ldots)\,T_3\,\frac{dx}{\sqrt{1-x^2}} \stackrel{!}{=} \frac{2}{\pi}\left(4\int_{-1}^{1}\sin(\ldots)\,x^3\frac{dx}{\sqrt{1-x^2}} - 3\int_{-1}^{1}\sin(\ldots)\frac{x\,dx}{\sqrt{1-x^2}}\right)$

$\stackrel{!}{=} \frac{2}{\pi}(0-0) = 0$

(Sol. 7 cont.)

9 (cont.) a) $P_a = 0.472001 - 0.499403\,(2x^2-1)$

$= 0.472001 - 0.499403\,\left(2\left(\frac{2}{\pi}t-1\right)^2-1\right)$

b) $P_b = b_0 + b_1 P_1 + b_2 P_2 + b_3 P_3$

$b_0 = \frac{1}{2}\int_{-1}^{1} \sin\left((x+1)\frac{\pi}{2}\right)\cdot 1\,dx \overset{!}{=} \frac{1}{2}\cdot\frac{4}{\pi} = \frac{2}{\pi}$

$b_1 = \frac{3}{2}\int_{-1}^{1} \sin(\dots)\,x\,dx \overset{!}{=} 0$

$b_2 = \frac{5}{2}\left(\int_{-1}^{1} \sin(\dots)\frac{3}{2}x^2\,dx - \int_{-1}^{1}\sin(\dots)\frac{1}{2}\,dx\right) =$

$= \frac{15}{\cancel{4}}\left(\frac{\pi^2-8}{\pi^3}\right)\cancel{4} - \frac{\cancel{2}\,5}{\pi}$

$b_3 = \frac{7}{2}\left(\frac{5}{2}\int_{-1}^{1}\sin(\dots)\,x^3\,dx - \frac{3}{2}\int_{-1}^{1}\sin(\dots)\,x\,dx\right) = 0$

Thus $P_b = \frac{2}{\pi} + \frac{1\overset{0}{\cancel{3}}\pi^2-120}{\pi^3}\left(P_2(x) = \frac{3}{2}x^2-\frac{1}{2}\right)\Big|_{x=\frac{2}{\pi}t-1}$

c) Chebyshew $a_4 = \frac{2}{\pi}\int_{-1}^{1}\sin(\dots)\overbrace{(8x^4-2x^2+1)}^{T_4}\frac{dx}{\sqrt{1-x^2}} \overset{!}{=} \dots$

≈ 0.0279921 (© via hints)

$|Error| \overset{!}{\approx} |a_4 T_4| \lesssim |a_4|\cdot 1 \approx 0.0279921$

Legendre $b_4 = \frac{9}{2}\int_{-1}^{1}\sin(\dots)\,P_4\,dx = \dots$

$\dots = \frac{9}{2}\left(\int_{-1}^{1}\sin(\dots)\frac{35}{8}x^4\,dx - \int_{-1}^{1}\sin(\dots)\frac{30}{8}x^2\,dx + \frac{3}{8}\int_{-1}^{1}\sin(\dots)\,dx\right) =$

$\dots \overset{!}{\approx} 0.051779$ (© via hints)

$\Rightarrow |Error| \overset{!}{\approx} |b_4 P_4| \lesssim |b_4|\cdot 1 \approx 0.051779$

(Sol. 7 cont.)

9 (cont.) d) As in 7 or 8 d)

10 $$y = e^x \overset{!}{\approx} \underbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}}_{\text{This is the 4th order Taylor approx.}} \quad (-1 \le x \le 1)$$

a) $$y = T_0 + T_1 + \frac{1}{2}\left(\frac{T_2 + T_0}{2}\right) + \frac{1}{6}\left(\frac{T_3 + 3T_1}{4}\right) + \frac{1}{24}\left(\frac{T_4 + 4T_2 + 3T_0}{8}\right) = \dots = \frac{81}{64}T_0 + \frac{9}{8}T_1 + \frac{13}{48}T_2 + \frac{1}{24}T_3$$

$$p = \frac{81}{64}\overset{(1)}{T_0} + \frac{9}{8}x + \frac{13}{48}(2x^2 - 1)$$

b) $$|\text{Error}| \approx \left|\frac{1}{24}T_3\right| \lessapprox \frac{1}{24}\cdot 1$$

c) Typically for Chebyshev approximation the error is uniformly small ("equal ripple")

d) Chebyshev as in c)

Taylor: Worse near the boundary. Very good near the midpoint. Not "equal ripple" (uniform).

Chebyshev seems better than 2nd order Taylor approximation.