Solutions 8

1. a) Degree $= 1 \Rightarrow$ Basis $\{1, x, y\}$

b) Design Matrix

$$\begin{array}{cc|ccc} & (x,y) & 1 & x & y \\ \hline A & (1,0) & & & \\ B & (0,1) & & & \\ C & (0,2) & & & \\ D & (1,3) & & & \end{array} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 3 \end{pmatrix}_{(4\times 3)} = G$$

Normal equations $G^t G a/ = G^t z/ \Leftrightarrow$

$$\Leftrightarrow \underbrace{\begin{pmatrix} 4 & 2 & 6 \\ 2 & 2 & 3 \\ 6 & 3 & 14 \end{pmatrix}}_{3\times 3} \begin{pmatrix} a_{00} \\ a_{10} \\ a_{01} \end{pmatrix} = \begin{pmatrix} 1\cdot 0 + 1\cdot 0 + 1(-1) + 1\cdot 1 \\ 1\cdot 0 + 0\cdot 0 + 0(-1) + 1\cdot 1 \\ 0\cdot 0 + 1\cdot 0 + 2(-1) + 3\cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

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c) $\Leftarrow \Rightarrow \quad a_{00} = -0.8 ; \quad a_{10} = 1 ; \quad a_{01} = 0.2$

Plane $= -0.8 + 1x + 0.2y$

2 a) Cf script and notebook mentioned in the problem text.

b) Ratio $= \dfrac{3.13397 \cdot 10^8}{4.50267} \approx 6.96025 \cdot 10^7$

The squared ratio and thus the condition number are rel. huge. The normal equations are ill-posed and the solution is unstable, i.e. the relative error of the solution with respect to the rel. error of the data (z-coord.) is "diverging".

The condition number has order $\sim 10^{14}$ (!)

The basis is $\{1, x, y, x^2, 2xy, y^2, x^3, 3x^2y, 3y^2x, y^3\}$.

(Sol. 8 cont.)

3 The basis is normalized by the substitutions
$x \to \frac{x-\mu_x}{\sigma_x}$ and $y \to \frac{y-\mu_y}{\sigma_y}$

a) The solution function for the least-squares approximation in expanded form is the same as in Problem 2a)

The diagonal matrix D in the singular value decomposition is (first 10 rows):

$$\begin{pmatrix} 235.07 & & & & & & \\ & 220.228 & & & & 0 & \\ & & 171.286 & & & & \\ & & & \ddots & & & \\ & 0 & & & 24.0546 & \\ & & & & & 21.5343 \end{pmatrix}$$

b) Ratio $= \frac{|\text{max. singular value} = 235.07|}{|\text{min. singular value} = 21.5343|}$

≈ 10.9161 (!)

The condition number is of order 10^2 (rel. small).

Therefore, the normalized basis guarantees numeric stability of the least-squares solution.