

Solutions 8

1 a) Degree = 1  $\Rightarrow$  Basis  $\{1, x, y\}$

b) Design Matrix

$$\begin{array}{c|ccc} (x, y) & 1 & x & y \\ \hline \textcircled{A} (1, 0) & 1 & 1 & 0 \\ \textcircled{B} (0, 1) & 1 & 0 & 1 \\ \textcircled{C} (0, 2) & 1 & 0 & 2 \\ \textcircled{D} (1, 3) & 1 & 1 & 3 \end{array} = G \quad (4 \times 3)$$

Normal equations  $G^T G a = G^T z \Leftrightarrow$

$$\Rightarrow \begin{pmatrix} 4 & 2 & 6 \\ 2 & 2 & 3 \\ 6 & 3 & 14 \end{pmatrix} \begin{pmatrix} a_{00} \\ a_{10} \\ a_{01} \end{pmatrix} = \begin{pmatrix} 1 \cdot 0 + 1 \cdot 0 + 1(-1) + 1 \cdot 1 \\ 1 \cdot 0 + 0 \cdot 0 + 0(-1) + 1 \cdot 1 \\ 0 \cdot 0 + 1 \cdot 0 + 2(-1) + 3 \cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$3 \times 3$

c)  $\Rightarrow a_{00} = -0.8; a_{10} = 1; a_{01} = 0.2$

Plane =  $-0.8 + 1x + 0.2y$

2

a) Cf script and notebook mentioned in the problem text.

b) Ratio =  $\frac{3.13397 \cdot 10^8}{4.50267} \approx \underline{\underline{6.96025 \cdot 10^7}}$

The squared ratio and thus the condition number are rel. huge. The normal equations are ill-posed and the solution is unstable, i.e. the relative error of the solution with respect to the rel. error of the data (z-coord.) is "diverging".

The condition number has order  $\sim 10^{14}$  (!)

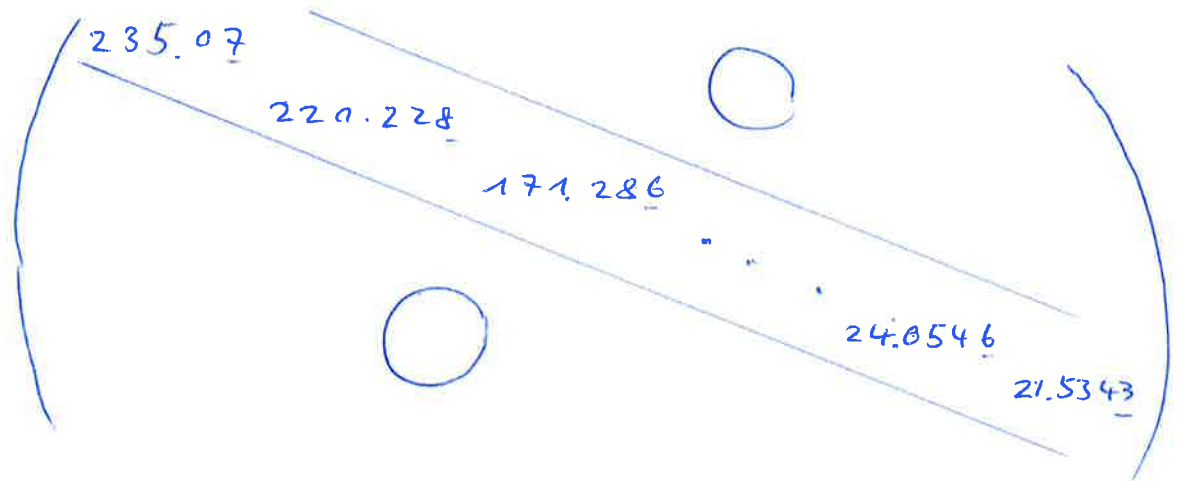
The basis is  $\{1, x, y, x^2, 2xy, y^2, x^3, 3x^2y, 3y^2x, y^3\}$ .

(Sol. 8 cont.)

3 The basis is normalized by the substitutions  
 $x \rightarrow \frac{x - \mu_x}{\sigma_x}$  and  $y \rightarrow \frac{y - \mu_y}{\sigma_y}$

a) The solution function for the least-squares approximation in expanded form is the same as in Problem 2a)

The diagonal matrix  $D$  in the singular value decomposition is (first 10 rows):



b) Ratio =  $\frac{|\max. \text{ singular value} = 235.07|}{|\min. \text{ " } = 21.5343|}$   
 $\approx \underline{10.9161} (!)$

The condition number is of order  $10^2$  (rel. small).

Therefore, the normalized basis guarantees numeric stability of the least-squares solution.