Exercises 11: Numeric Ordinary Differential Equations I

The problems are solvable with a computer, normally. There are exceptions.
The symbol **|** means „or“, the symbol ***** „optional“, the symbol ****** „optional and advanced“ and the symbol © means that a computer is required or helpful.

1. **a)** © Solve the initial value problem $y' = x\,y^{1/3}$ $\quad y(1) = 1$ numerically by the method of Taylor with order $p = 4$ and fixed step-size $h = 0.1$ for the x-values 1.1 und 1.2 (two steps). All final (!) results should be rounded to the 10th digit.

b) © Compute the local error (slope) $\tau_h(x_n)$ and from this $h \cdot \tau_h(x_n)$ as well as the global error for the two steps in a).

Hints: The exact solution of the equation is $y = \left(\dfrac{x^2+2}{3}\right)^{3/2}$.

Compute the derivations $y', y'', y''', y^{(4)}$ algebraically from the differential equation by iteratively applying the product and chain rule (no computer required), as:

$$y'' = \left(x\,y^{1/3}\right)' = y^{1/3} + x\left(y^{1/3}\right)' = y^{1/3} + x\left(\frac{1}{3}y^{-2/3}\cdot y'\right) =$$

$$= y^{1/3} + x\left(\frac{1}{3}y^{-2/3}\cdot x\,y^{1/3}\right) = y^{1/3} + \frac{1}{3}x^2 y^{-1/3}$$

$$y''' = \left(y^{1/3} + \frac{1}{3}x^2 y^{-1/3}\right)' = \frac{1}{3}y^{-2/3}\cdot y' + 2x\frac{1}{3}y^{-1/3} - \frac{1}{9}x^2 y^{-4/3}\cdot y' = \cdots = x\,y^{-1/3} - \frac{1}{9}x^3 y^{-1}$$

$$y^{(4)} = \left(x\,y^{-1/3} - \frac{1}{9}x^3 y^{-1}\right)' = \cdots = 1y^{-1/3} - \frac{2}{3}x^2 y^{-1} + \frac{1}{9}x^4 y^{-5/3}$$

2. © Compute the coordinates of the (t, φ) point after the first step rounded to the 6th digit in:

a) Example 1.2 / Figure 1.2b on page 4/5 (script), Taylor-Method of order $p = 1$ (Euler)

b) Example 1.3 / Figure 1.3b on page 6/7 (script), Taylor-Method of order $p = 2$

Finally, compute the local error (of slope) $\tau_h(t_n)$ and from this $h \cdot \tau_h(t_n)$ as well as the global error for ab).

Hints: A reference value for the exact φ is 0.28374<u>7</u> (rounded to the 6th digit). The examples refer to the differential equation $\varphi' = c\left(1 - \varepsilon\cos\varphi\right)^2$ $\quad \varphi(0) = 0$ with $c = 1$ and $\varepsilon = 0.25$ (angle-velocity-relation of an elliptical gravitational orbit).