Lösungen Chapter 19

19.9 (2nd ed.) Herleitung Bedingungen Ordnung 3

Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & & \\ c_2 & a_{21} & 0 & \\ c_3 & a_{31} & a_{32} & 0 \\ \hline & b_1 & b_2 & b_3 \end{array}$$

$$\left.\begin{aligned} k_1 &= h f(x,y) \\ k_2 &= h f(x + c_2 h,\ y + a_{21} k_1) \\ k_3 &= h f(x + c_3 h,\ y + a_{31} k_1 + a_{32} k_2) \end{aligned}\right\} \text{3 stages}$$

$$y(x+h) = y(x) + b_1 k_1 + b_2 k_2 + b_3 k_3$$

⊳ Taylor: $y(x+h) - y(x) = hf + \frac{1}{2} h^2 \underbrace{(f_x + f_y f)}_{F_1} +$

$$+ \frac{1}{6} h^3 \big(\underbrace{f_{xx} + 2f f_{xy} + f^2 f_{yy}}_{F_2} + \underbrace{f_x f_y + f f_y^2}_{+\ f_y F_1} \big) + \dots$$

⊳ Taylor (multi-variate)

$$k_1 = hf$$

$$k_2 = h\Big(f + f_x c_2 h + f_y a_{21} h f + \frac{1}{2!} f_{xx} c_2^2 h^2 + \frac{1}{1!1!} f_{xy} c_2 a_{21} f h^2 + \frac{1}{2!} f_{yy} a_{21}^2 f^2 h^2 + \dots\Big)$$

$$k_3 = h\Big(f + f_x c_3 h + f_y (a_{31} k_1 + a_{32} k_2) + \frac{1}{2!} f_{xx} c_3^2 h^2 + \frac{1}{1!1!} f_{xy} c_3 h (a_{31} k_1 + a_{32} k_2) + \frac{1}{2!} f_{yy} (a_{31} k_1 + a_{32} k_2)^2 + \dots$$

$$\stackrel{!}{=} h\Big(f + f_x c_3 h + f_y a_{31} h f + f_y a_{32} (hf + f_x c_2 h^2 + f_y a_{21} h^2 f + \dots) + \frac{1}{2!} f_{xx} c_3^2 h^2 + \frac{1}{1!1!} f_{xy} c_3 h^2 (a_{31} f + a_{32} f + \dots) + \frac{1}{2!} f_{yy} f^2 h^2 \cdot (a_{31} + a_{32} + \dots)^2 + \dots\Big)$$

$$= h\Big(f + (f_x c_3 + f_y f a_{31} + f_y f a_{32}) h + \Big\{ f_x f_y c_2 a_{32} + f_y^2 f a_{32} a_{21} + \frac{1}{2!} f_{xx} c_3^2 + \frac{1}{1!1!} f_{xy} c_3 f (a_{31} + a_{32}) + \frac{1}{2!} f_{yy} f^2 (a_{31} + a_{32})^2 \Big\} h^2 + \dots\Big)$$

Abkürzung: $\{\dots\} =: (*)$

(Lösungen Chapter 19 cont.)

19.9 (cont.) Koeffizientenvergleich h, h^2, h^3

$$y(x+h) - y(x) = b_1 k_1 + b_2 k_2 + b_3 k_3$$

$$\Longrightarrow (b_1+b_2+b_3) h f = 1 h f \Longrightarrow b_1 + b_2 + b_3 = 1$$

$$\Longrightarrow b_2 (f_x c_2 + a_{21} f f_y) + b_3 (f_x c_3 + f_y f(a_{31} + a_{32})) h^2 = \frac{1}{2} F_1 h^2$$

$$\Longrightarrow \left(b_2 \left(\frac{1}{2!} f_{xx} c_2^2 + \frac{1}{1!1!} f_{xy} c_3 a_{21} f + \frac{1}{2!} f_{yy} a_{21} f^2\right) + b_3 (1)\right) h^3 = \frac{1}{6} h^3 (F_2 + f_y F_1)$$

(1) Daraus

Via f_x : $b_2 c_2 + b_3 c_3 = \frac{1}{2}$

Via $f \cdot f_y$: $b_2 a_{21} + b_3 (a_{31} + a_{32}) = \frac{1}{2}$

Via f_{xx} : $\frac{1}{2} b_2 c_2^2 + \frac{1}{2} b_3 c_3^2 = \frac{1}{6}$

Via $f_{xy} f$: $b_2 c_2 a_{21} + c_3 (a_{31} + a_{32}) b_3 = \frac{2}{6}$

$\rightarrow$ mit (1) $c_3 = a_{31} + a_{32}$ (2)

Via $f^2 f_{yy}$: $\frac{1}{2} a_{21}^2 b_2 + \frac{1}{2} b_3 (a_{31} + a_{32})^2 = \frac{1}{6}$

Via $f_x \cdot f_y$: $b_3 c_2 a_{32} = \frac{1}{6}$ / Via $f_y^2 f$: $b_3 a_{32} a_{21} = \frac{1}{6}$

$\rightarrow b_3 \neq 0 \rightarrow a_{32} \neq 0 \rightarrow c_2 = a_{21}$ (1)

(1) und (2) sind Zeilenbedingungen (Butcher tableau).

$\rightarrow$ Reduktion:

$b_1 + b_2 + b_3 = 1$

$b_2 c_2 + b_3 c_3 = \frac{1}{2}$ $\qquad c_2 = a_{21}$ (1)

$\frac{1}{2} b_2 c_2^2 + \frac{1}{2} b_3 c_3^2 = \frac{1}{6}$ $\qquad c_3 = a_{31} + a_{32}$ (2)

$b_3 c_2 a_{32} = \frac{1}{6}$

Beispiel-Lösung

$b_1 = \frac{1}{6}$; $b_2 = \frac{4}{6}$; $b_3 = \frac{1}{6}$

$c_2 = \frac{1}{2}$; $c_3 = 1$; $a_{21} = \frac{1}{2}$; $a_{32} = 2$; $a_{31} = -1$ ✓