**Exercises 13: Numeric Ordinary Differential Equations II / III**

The problems are solvable without a computer, normally. There are exceptions.
The symbol **|** means „or“, the symbol * „optional“, the symbol ** „optional and advanced“ and the symbol © means that a computer is required or helpful.

**1.** © Solve the initial value problem $\varphi' = c\left(1-\varepsilon\cos\varphi\right)^2 \quad \varphi(0)=0$ with $c$ = 1 and $\epsilon$ = 0.25 numerically by applying the Heun-Euler 2(1) embedded adaptive method with classical step-size control until 3 proceeding steps are executed. The initial step-size equals 0.001, the accuracy goal (ag) 4 and the precision goal is 4, either.

Create a table listing values for ($x$, $y$, $h$, $e_k$, $\left(\frac{\|e_k\|}{\varepsilon}\right)^{-\frac{1}{\tilde{p}}}$, $h_{new}$, *state*) containing at least three proceeding steps.

*Hints*: The problem refers to Example 1.8 (angle-velocity-relation of an elliptical gravitational orbit,script p. 19). The order is $\tilde{p}=2$ .

**2.** Following the same lines as in Example 1.11 (script p.27/28) analyze the absolute A-stability (Dahlquist stability) for the "midpoint" method:

$$
\begin{aligned}
&y_0 = y(x_0)\\
&y_{k+1} = y_k + h_k\left(f\left(x_k+\frac{1}{2}h_k, y_k+\frac{1}{2}h_k f\left(x_k,y_k\right)\right)\right)
\end{aligned}
$$

Derive the A-stability polynomial and plot the A-stability region. What are the boundaries of the stability interval on the negative real axis?

**3.** Using Theorem 1.3 in the script (p. 29) gain the A-stability polynomial $F_1(z)=1+z+\frac{z^2}{2}+\frac{z^3}{6}$ for the embedded adaptive method SS3(2) with Butcher tableau

| | | | | |
|---|---|---|---|---|
| $0$ | | | | |
| $\frac{1}{2}$ | $\frac{1}{2}$ | | | |
| $1$ | $-1$ | $2$ | | |
| $1$ | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{1}{6}$ | |
| | $\frac{1}{6}$ | $\frac{2}{3}$ | $\frac{1}{6}$ | $0$ |
| | $\frac{1}{72}\left(\sqrt{82}-10\right)$ | $\frac{1}{36}\left(10-\sqrt{82}\right)$ | $\frac{1}{144}\left(28-\sqrt{82}\right)$ | $\frac{1}{48}\left(\sqrt{82}-16\right)$ |

**4.** Given the differential equation $y' = \underbrace{-\sqrt{x^2 + y^2}}_{f(x,y)}$ $\quad y(0) = 4$ carry out a stiffness detection test using the A-stability region and the partial derivative $f_y$ at the initial values and stepsize $h$ = 1. The method is defined by the Butcher tableau below.

| | | |
|---|---|---|
| 0 | | |
| 1/2 | 1/2 | |
| *b* | 0 | 1 |
| $\hat{b}$ | 1 | 0 |