Exercises 14: Numeric Ordinary Differential Equations IV

The problems are solvable without a computer, normally. There are exceptions.
The symbol **|** means „or“, the symbol ***** „optional“, the symbol ****** „optional and advanced“ and the symbol © means that a computer is required or helpful.

1. Solve the 2d linear ODE-system below approximately by expanding Taylor series for $x(t)$ and $y(t)$ up to order 3:

$$\begin{aligned} x' &= -x-y \\ y' &= x-y \qquad x(0)=1, \quad y(0)=0 \end{aligned}$$

Hints: $x(t) \approx x(0) + x'(0)t + \frac{x''(0)}{2}t^2 + \frac{x'''(0)}{6}t^3$, e.g. The derivatives are gained iteratively:

$x'(0) = -x(0) - y(0) = -1 - 0 = -1$, then

$x''(0) = -x'(0) - y'(0) = 1 - (x-y) = 1 - (1-0) = 0$ a.s.o.

2. The acceleration equations below describe the trajectory $(x(t), y(t), z(t))$ in 3d space of a particle:

$$\begin{aligned} x'' &= f_1(t,x,y,z,x',y',z') \\ y'' &= f_2(t,x,y,z,x',y',z') \\ z'' &= f_3(t,x,y,z,x',y',z') \end{aligned}$$

Write down the equivalent ODE-system of 1st order. What is its dimension?

3. Transform the 2nd order scalar ODE $y'' = f(x,y,y') \quad y(x_0) = y_0,\ y'(x_0) = y_1$ into a first-order ODE-system taking into account the initial conditions.

4. © Solve the van-der-Pol ODE-system $\begin{pmatrix} z' = v \\ v' = \mu(1-z^2)v - z \end{pmatrix}$ $z(0)=1, v(0)=-1, \quad \mu = 0.2$ numerically by applying the Heun-Euler 2(1) embedded adaptive method with classical step-size control until 3 proceeding steps are executed. The initial step-size equals 0.001, the accuracy goal (ag) 1 and the precision goal is 2.

Create a table listing values for (t, $\{z, v\}$, h, e_k, $\left\| \frac{\vec{e}_n}{\varepsilon_a + \varepsilon_r \vec{y}_n} \right\|^{-1/\tilde{p}}$, h_{new}, *state*) containing at least three proceeding steps.

Hints: The problem refers to Example 2.4 (script p. 41ff). The order is $\tilde{p} = 2$. The expression $\left\| \dfrac{\vec{e}_n}{\varepsilon_a + \varepsilon_r \vec{y}_n} \right\|^{-1/\tilde{p}}$ is a short form for $\max\left\{ \dfrac{\left|\vec{e}_n^{(1)}\right|}{\varepsilon_a + \varepsilon_r \left|\vec{y}_n^{(1)}\right|}, \dfrac{\left|\vec{e}_n^{(2)}\right|}{\varepsilon_a + \varepsilon_r \left|\vec{y}_n^{(2)}\right|} \right\}^{-\frac{1}{\tilde{p}}}$.

5. Write down the classical Runge-Kutta scheme for the 2d ODE-system

$$
\begin{aligned}
y' &= f_1(x, y, p) \\
p' &= f_2(x, y, p) \qquad y(x_0) = y_0, \quad p(x_0) = p_0
\end{aligned}
$$

6. © Carry out numerically the first step with the classical Runge-Kutta scheme for the 2nd order differential equation (van der Pol) with step-size h = 0.2:

$$
y'' - 0.1(1 - y^2)y' + y = 0 \quad y(0) = 1, \; y'(0) = 0
$$