Exercises 1: Polynomial Interpolation I-II

The problems are solvable without a computer, normally. There are rare exceptions.
The symbol **|** means „or", the symbol ***** „optional", the symbol ****** „optional and advanced" and the symbol © means that a computer is required or helpful.

1. **[Schaum 8.6]** Compute the divided differences up to order 3 for the data:

x_k	0	1	2	4
y_k	1	1	2	5

2. **[Schaum 8.7]** Show by computation that $y(x_0, x_1) = y(x_1, x_0)$ for first order divided differences.

3. **[Schaum 8.8]** Prove the symmetry of the 2nd order divided difference $y(x_0, x_1, x_2)$ by logical reasoning.
Hint: The second-order difference $y(x_0, x_1, x_2)$ is the leading coefficient (number) of the power x^2 in the interpolating parabola p of the three (x, y)-data points $\{(x_0, y_0), (x_1, y_1), (x_2, y_2)\}$.
Does p depend on the order of the three data points?

4. **a)** **[Schaum 8.23]** Compute the divided differences up to order 3 for the data …

x_k	0	1	4	6
y_k	1	−1	1	−1

b) **[Schaum 8.25]** … and then those for the data:

x_k	4	1	6	0
y_k	1	−1	−1	1

Check your computations against the symmetry statement in Exercise 3.

5. **[Schaum 8.29]** Carefully check the statements about the divided differences below by logical reasoning or short (!) computations for the specific model function
$y(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$ for fixed arguments $x_0, x_1, \ldots, x_n \in \mathbb{R}$.

a) $y(x_0, x_1, \ldots, x_p) = 0$ for $p = 0,1,\ldots,n$

b) $y(x_0, x_1, \ldots, x_n, x) = 1$ for any $x \in \mathbb{R} - \{x_0, x_1, \cdots, x_n\}$

c) $y(x_0, x_1, \ldots, x_n, x, z) = 0$ for any $x, z \in \mathbb{R} - \{x_0, x_1, \cdots, x_n\}$, $x \neq z$

Hints: The specific model function $y(x)$ corresponds to the Newton polynomial $\pi_{n+1}(x)$ and this is zero for $x = x_0, x = x_1, \ldots, x = x_p$. For **b)** consider x as a further argument different from $x_0, x_1, \ldots, x_n$. For **c)** or alternatively for b) use a formula representing divided differences as higher derivatives.

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6. **[Schaum 8.31]** For equally spaced arguments ($x_{k+1} - x_k = h =$ const.) develop the difference formula below by induction starting with the case k = 0, then going to k = 1, k = 2 a.s.o. :

$$y(x_0, x_1, \ldots, x_k) = \frac{\Delta^k y_0}{h^k k!}$$

The symbol Δ denotes the difference operator, thus $\Delta y_k = y_{k+1} - y_k$, and the formula Δ^m abbreviates the m-fold composition (nested application) of Δ.

7. © **[Schaum 2.9]** The model function $y(x) = \cos(\frac{1}{2}\pi\, x)$ has to be interpolated linearly within the arguments x = 0;1 by a polynomial $p(x)$. Estimate the absolute maximum error $\max\limits_{x\in[0,1]} |y(x) - p(x)|$ by a simple expression and check your estimation at the position x = ½ by comparing with the exact interpolation error.

8. © **[Schaum 2.14 – 2.18]** The model function $y(x) = \sin(\frac{1}{2}\pi\, x)$ has to be interpolated using the arguments x = 0; 1; 2 by a quadratic polynomial $p(x)$.

a) Give a formula for the interpolation error $y(x) - p(x)$ $(x \in [0,2])$ using higher derivatives.

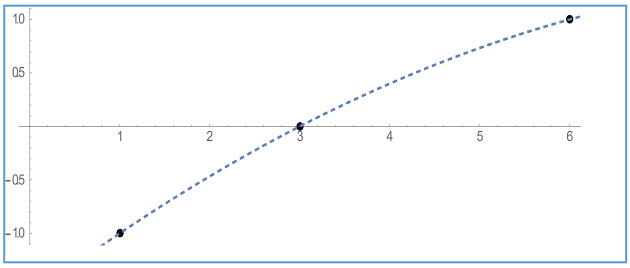
b) Estimate the absolute maximum error $\max\limits_{x\in[0,2]} |y(x) - p(x)|$ by a simple expression and check your estimation at the position x = ½ by comparing with the exact interpolation error.

c) Compare the derivatives $y'(x)$ und $p'(x)$ at the position x = ½.

d) Compare the second derivatives $y''(x)$ und $p''(x)$ at the position x = ½.

e) Compare the integrals $\int_0^2 y(x)dx$ und $\int_0^2 p(x)dx$.

9. Example to the **Aitken-Neville-Recursion formula**. Compute the polynomials $p_0 = y_0, p_1 = y_1, p_2 = y_2$ (constants) as well as p_{01}, p_{12} (straight lines) and p_{012} (interpolating parabola) for the (x, y)-data points $\{(1,-1),(3,0),(6,1)\}$.



Sketch the configuration including the polynomials $p_0, p_1, p_2, p_{01}, p_{12}$.