Exercises 4: Spline Interpolation I (Univariate Spline Interpolation)

The problems are solvable without a computer, normally. There are rare exceptions.
The symbol **|** means „or", the symbol ***** „optional", the symbol ****** „optional and advanced" and the symbol © means that a computer is required or helpful.

1. © The data below was generated by the sine function ($y = sin(x)$) :

x_i	0	$\pi/3$	$2\pi/3$	π
y_i	0	$\sqrt{3}/2$	$\sqrt{3}/2$	0

a) Compute the cubic natural spline interpolation S for the range [0, π].
b) Compute the cubic „clamped" spline interpolation S for the range [0, π].
c) For ab) give maximum estimations for the following error quantities $|\, y(x) - S(x)\,|$, $|\, y'(x) - S'(x)\,|$ and $|\, y''(x) - S''(x)\,|$.

2. Write down a system of linear equations for the c-coefficients of the cubic natural spline interpolation of the data listed below (but do not solve the system) :

x_i	0	2	2.5	3	3.5	4	4.5	5	6
y_i	0	2.9	3.5	3.8	3.5	3.5	3.5	2.6	0

3. **a)** Calculate the derivatives of order 0 up to 3 at the "left" argument positions x_i for the cubic spline patch $S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \qquad (i = 0, 1, \ldots, n-1)$

b) Calculate the derivatives of order 0 up to 3 at the "right" argument positions x_{i+1} for the cubic spline patch $S_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 \qquad (i = 0, 1, \ldots, n-1)$

4. **a)** The data below comes from one single straight line. Find (!) or compute the "cubic" natural spline interpolation in the range [-1, 3]. *Hint*: By logical reasoning computations can be reduced significantly.

x	-1	0	1	3
y	0	1	2	4

b) The data below comes from the parabola $y(x) = x^2$. Why does the cubic natural spline interpolation not reproduce the parabola ? No computations required here.

x	-1	0	1	3
y	1	0	1	9

c) What values will be expected for the c- and d-coefficients for a natural cubic spline interpolation if the data points $(x_i, y_i) \quad (i = 0,1,...,n)$ come from one single straight line?

Hint: A spline interpolation generally is unique. Consider the right hand side of the system of equations for a natural spline interpolation.

5. ** The natural cubic spline interpolation minimizes the mean total curvature $\int_{x_0}^{x_n} | f''(x) |^2 dx$ (a measure of "energy") among all interpolating C^2 functions f in the range $[x_0, x_n]$. This fact is called Holladay theorem and its proof is not an easy exercise at all.

Hint: Schaum's Outline of Numerical Analysis 2nd ed., Problem 9.17

6. ** The piecewise linear spline interpolation S minimizes the mean total slope $\int_{x_0}^{x_n} | f'(x) |^2 dx$ among all interpolating continuous function f in the range $[x_0, x_n]$ that are piecewise C^1 (per patch). The proof of this fact is not an easy exercise.

Hint: Repeat the reasoning in Schaum's Outline of Numerical Analysis 2nd ed., Problem 9.17, starting with the expression $\int_{x_0}^{x_n} | f'(x) |^2 dx - \int_{x_0}^{x_n} | S'(x) |^2 dx$.