Exercises 6: Linear Least-Squares Approximation (Discrete Data)

The problems are solvable without a computer, normally. There are exceptions.
The symbol **|** means „or“, the symbol ***** „optional“, the symbol ****** „optional and advanced“ and the symbol © means that a computer is required or helpful.

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1. © Compute a linear least-squares approximating parabola for the "second window" of five consecutive points (starting with $x = 2$) in the data of script Example 1.1:

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{{x,y}} = {{1,1.04},{2,1.37},{3,1.70},{4,2.00},{5,2.26},
           {6,2.42},{7,2.70},{8,2.78},{9,3.00},{10,3.14}}
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a) Write down the design matrix and the system of normal equations.

b) © Solve the linear system.

c) © Compute the output y and the derivative (!) of the approximation at the central coordinate ($x = 4$).

2. These problems generalize Example 1.1 from the script leading to a derivation filter of Savitzky-Golay type.

For a window of 5 consecutive points with uniform x – distance h

$\{\{x,y\}\} = \{\{x_{k-2},y_{k-2}\},\{x_{k-1},y_{k-1}\},\{x_k,y_k\},\{x_{k+1},y_{k+1}\},\{x_{k+2},y_{k+2}\}\}$

and central x-coordinate x_k a least-squares parabela $y = a_0 + a_1 t + a_2 t^2$ has to be computed; here t denotes an integer coordinate with center at 0:

$$t = \frac{x - x_k}{h} \quad (t = -2,-1,0,1,2).$$

a) Write down the design matrix and the system of normal equations.

b) Algebraically solve the normal system for the coefficients a_0 , a_1 , a_2 .

c) Approximately compute y as well as the first derivative of the data at the central coordinate $x = x_k$ (*Hint*: $x = x_k$ corresponds to $t = 0$).

d) Approximately compute y and the first derivative for $x = x_{k-2}$ (this is the left side).

3. © Apply the filter formulas developed in the problems 2c above for the data to compute approximately y for $x = 3, \ldots, 8$.

x_k	1	2	3	4	5	6	7	8	9	10
y_k	1.04	1.37	1.70	2.00	2.26	2.42	2.70	2.78	3.00	3.14

4. © Solve problem 1 again by using the orthogonal polynomials $\{p_{k,N}(t)\}_{k=0,\ldots,2}$ (cf. script example 1.4). Compare the results with those of problem 1.

5. Examine and compute a least-squares approximative quadratic parabola for the data

x	-2	-1	0	1	2
y	0	1	2	3	1

with respect to the basis functions $\left\{1, -\frac{x}{2}, \frac{x^2}{2} - 1\right\}$ in the following sense:

a) Compute the design matrix G and the normal matrix. *Hint*: The normal matrix here is diagonal!

b) Solve the system of normal equations and write down a formula for the approximating parabola.

c) What are the dimensions of the unitary matrices U, V, as well as the diagonal matrix D, in the singular value decomposition $G = U D V^{tr}$?

d) What are the entries (singular values) in the matrix D of c).

e) Give three orthogonal basis polynomials (with respect to the data given) as formulas in the variable x.