Exercises 7: Linear Least-Squares Approximation (cont.)

The problems are solvable without a computer, normally. There are exceptions.
The symbol **|** means „or“, the symbol ***** „optional“, the symbol ****** „optional and advanced“ and the symbol © means that a computer is required or helpful.

1. The Chebyshev Polynomials $T_n(x)$ for $-1 \le x \le 1$ and $n \in \mathbb{N}_0$ are defined by the trigonometric expression $T_n(x) = \cos\left(n \cdot \arccos x\right)$.

By trigonometric expansion develop formulas for $T_0(x)$, $T_1(x)$, $T_2(x)$ and $T_3(x)$.

Hint: $\cos\left(2a\right) = 2\cos^2 a - 1$ and $\cos\left(3a\right) = 4\cos^3 a - 3\cos a$

2. Use the recursion formula $T_{n+1}(x) = 2x \cdot T_n(x) - T_{n-1}(x) \quad (n \in \mathbb{N})$ to compute $T_4(x)$ and $T_5(x)$ from the results from Problem 1 above.

3. Express each of the monomials $1, x, x^2, x^3, x^4, x^5$ as linear combinations of the Chebyshev polynomials $T_0(x), T_1(x), T_2(x), T_3(x), T_4(x), T_5(x)$.

Hint: Begin with 1, x … and continually use the results already obtained.

The next three problems are parallel to Problems 1 – 3 above.

4. The Legendre Polynomials $P_n(x)$ for $n \in \mathbb{N}_0$ are defined by the by the n-th-order derivative expression $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left(x^2 - 1\right)^n$ (Rodriguez formula).

Compute $P_0(x), P_1(x)$ and $P_2(x)$ from this formula.

5. Use the Bonnet recursion formula $(n+1)P_{n+1}(x) = (2n+1)x \cdot P_n(x) - n \cdot P_{n-1}(x) \quad (n \in \mathbb{N})$ to compute $P_3(x)$ and $P_4(x)$ from the results from Problem 4 above.

6. Express each of the monomials $1, x, x^2, x^3, x^4$ as linear combinations of the Legendre polynomials $P_0(x), P_1(x), P_2(x), P_3(x), P_4(x)$.

Hint: Begin with 1, x … and continually use the results already obtained.

7. Compute continuously approximating least-squares lines for the model function $y(x) = x^3 \quad (-1 \le x \le 1)$ by

a) Chebyshev approximation with the weight function $w(x) = 1/\sqrt{1-x^2} \quad (-1 < x < 1)$.

b) Legendre approximation with the weight function $w(x) = 1$.

c) Estimate the maximum approximating errors in ab) by the coefficients of the Chebyshev polynomial $T_2(x)$ and Legendre polynomial $P_2(x)$, respectively.

d) © By examination of graphical plots try to compare and describe the quality of approximation errors (overall, middle, boundary).

Hints: Use the truncation method based on the results from Problem 3 and 6, respectively.

8. Again compute continuously approximating least-squares lines for the model function $y(t) = t^2 \quad (0 \le t \le 1)$ by

a) Chebyshev approximation with the weight function $w(x) = 1/\sqrt{1-x^2} \quad (-1 < x < 1)$.

b) Legendre approximation with the weight function $w(x) = 1$.

c) Estimate the maximum approximating errors in ab) by the coefficients of the Chebyshev polynomial $T_2(x)$ and Legendre polynomial $P_2(x)$, respectively.

d) © By examination of graphical plots try to compare and describe the quality of approximation errors (overall, middle, boundary) in ab).

Hints: Use the truncation method referred to in Problem 7 above. Note the non-standard range of the variable t, here.

9. Compute continuously approximating least-squares cubic polynomials for the model function $y(t) = \sin(t) \quad (0 \le t \le \pi)$ by

a) © exact Chebyshev approximation with the weight function $w(x) = 1/\sqrt{1-x^2} \quad (-1 < x < 1)$.

b) © exact Legendre approximation with the weight function $w(x) = 1$.

c) © Estimate the maximum approximating errors in ab) by the coefficients of the Chebyshev polynomial $T_4(x)$ and Legendre polynomial $P_4(x)$, respectively.

d) © By examination of graphical plots try to compare and describe the quality of approximation errors (overall, middle, boundary).

Hints: Note the non-standard range of the variable t, here. Finally, use the integrals listed below:

$$\int_{-1}^{1} \sin\left((x+1)\frac{\pi}{2}\right) \cdot \frac{1}{\sqrt{1-x^2}}\, dx = \pi \cdot BesselJ(0, \frac{\pi}{2}) \approx 1.4828\underline{4}$$

$$\int_{-1}^{1} x \cdot \sin\left((x+1)\frac{\pi}{2}\right) \cdot \frac{1}{\sqrt{1-x^2}}\, dx = 0$$

$$\int_{-1}^{1} x^2 \cdot \sin\left((x+1)\frac{\pi}{2}\right) \cdot \frac{1}{\sqrt{1-x^2}}\,dx = 2BesselJ\left[1,\frac{\pi}{2}\right] - \pi BesselJ\left[2,\frac{\pi}{2}\right] \approx 0.34918\underline{7}$$

$$\int_{-1}^{1} x^3 \cdot \sin\left((x+1)\frac{\pi}{2}\right) \cdot \frac{1}{\sqrt{1-x^2}}\,dx = 0$$

$$\int_{-1}^{1} x^4 \cdot \sin\left((x+1)\frac{\pi}{2}\right) \cdot \frac{1}{\sqrt{1-x^2}}\,dx = -\frac{\left(-12+\pi^2\right) BesselJ\left[2,\frac{\pi}{2}\right]}{\pi} \approx 0.16932\underline{9}$$

$$\int_{-1}^{1} \sin\left((x+1)\frac{\pi}{2}\right) dx = \frac{4}{\pi}$$

$$\int_{-1}^{1} x \cdot \sin\left((x+1)\frac{\pi}{2}\right) dx = 0$$

$$\int_{-1}^{1} x^2 \cdot \sin\left((x+1)\frac{\pi}{2}\right) dx = \frac{4\left(-8+\pi^2\right)}{\pi^3}$$

$$\int_{-1}^{1} x^3 \cdot \sin\left((x+1)\frac{\pi}{2}\right) dx = 0$$

$$\int_{-1}^{1} x^4 \cdot \sin\left((x+1)\frac{\pi}{2}\right) dx = \frac{4\left(384-48\pi^2+\pi^4\right)}{\pi^5}$$

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10. Compute a continuously approximating least-squares parabola for the model function
$y(x) = e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \quad (-1 \le x \le 1)$ by

a) Chebyshev approximation with the weight function $w(x) = 1 / \sqrt{1-x^2} \quad (-1 < x < 1)$.

b) Estimate the maximum approximating error in a) by the coefficient of the Chebyshev polynomial $T_3(x)$.

c) © By examination of graphical plots try to describe the quality of the approximation error (overall, middle, boundary).

d) © By examination of graphical plots try to compare and describe the quality of the approximation errors (overall, middle, boundary) of the 2nd order Taylor approximation $e^x \approx 1 + x + \frac{x^2}{2}$ and the Chebyshev approximation.

Hints: As in Problem 7 use the truncation method based on the results from Problem 3 and 6, respectively. Apply this method to the polynomial Taylor approximation
$y(x) = e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \quad (-1 \le x \le 1)$.