Minmax Approximation

Methods for discrete data

1 Uni-variate linear minmax approximation

In discrete minmax approximation (a.k.a. Chebyshev-approximation) the main objective is to approximate discrete data with a model function such that the worst-case = maximum absolute error becomes minimal, i.e. the value of the global maximum absolute error $|e|$ in Fig. 1.1 is minimal.

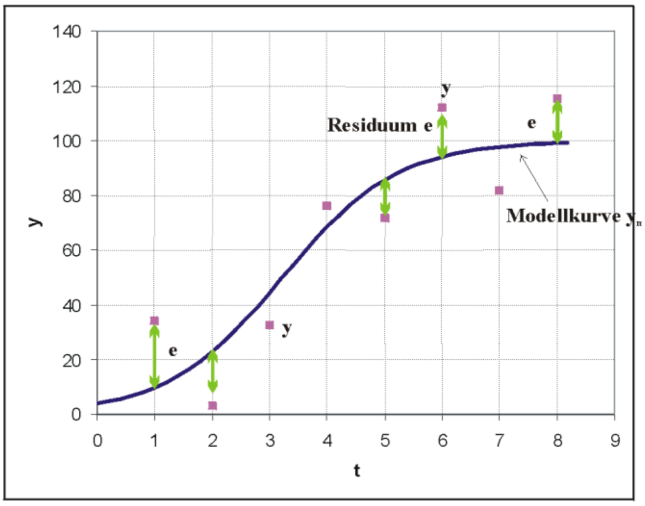

Figure 1.1: Discrete data of 8 points approximated by a modelling cubic curve with 8 residuals e.

Given discrete data (e.g. by a measurement) $(x_0, y_0), (x_1, y_1)_1, \ldots, (x_N, y_N) = \{(x_i, y_i)\}_{i=0,\ldots,N}$

or (in the multi-variate arguments case) $(\vec{x}_0, y_0), (\vec{x}_1, y_1)_1, \ldots, (\vec{x}_N, y_N) = \{(\vec{x}_i, y_i)\}_{i=0,\ldots,N}$

and a set of modelling basis functions $g_0, g_1, \ldots, g_m = \{g_j\}_{j=0,\ldots,m}$ the components of the residual vector $\vec{r} = (r_i)_{i=0,\cdots,N}$ are defined as the differences between the data y-values and the y-output values of the linearly combined model function $\sum_{j=0}^{m} a_j g_j(x)$ evaluated at the data x-arguments:

$$r_i = y_i - \sum_{j=0}^{m} a_j g_j(x_i) \quad (i = 0, \ldots, N) \tag{1.1}$$

The maximum of the absolute residuals, a.k.a. as the maximum-norm $\|\cdots\|_\infty$, has to be minimized:

$$\gamma := \max_{i=0,\ldots,N} |r_i| =: \|\vec{r}\|_\infty \quad \rightarrow \quad \min! \tag{1.2a}$$

Example 1.1: Minmax parabola

The 5-point data `{ {3,1.70},{4,2.00},{5,2.26},{6,2.42},{7,2.70} }` is to be approximated by a minmax-parabola.

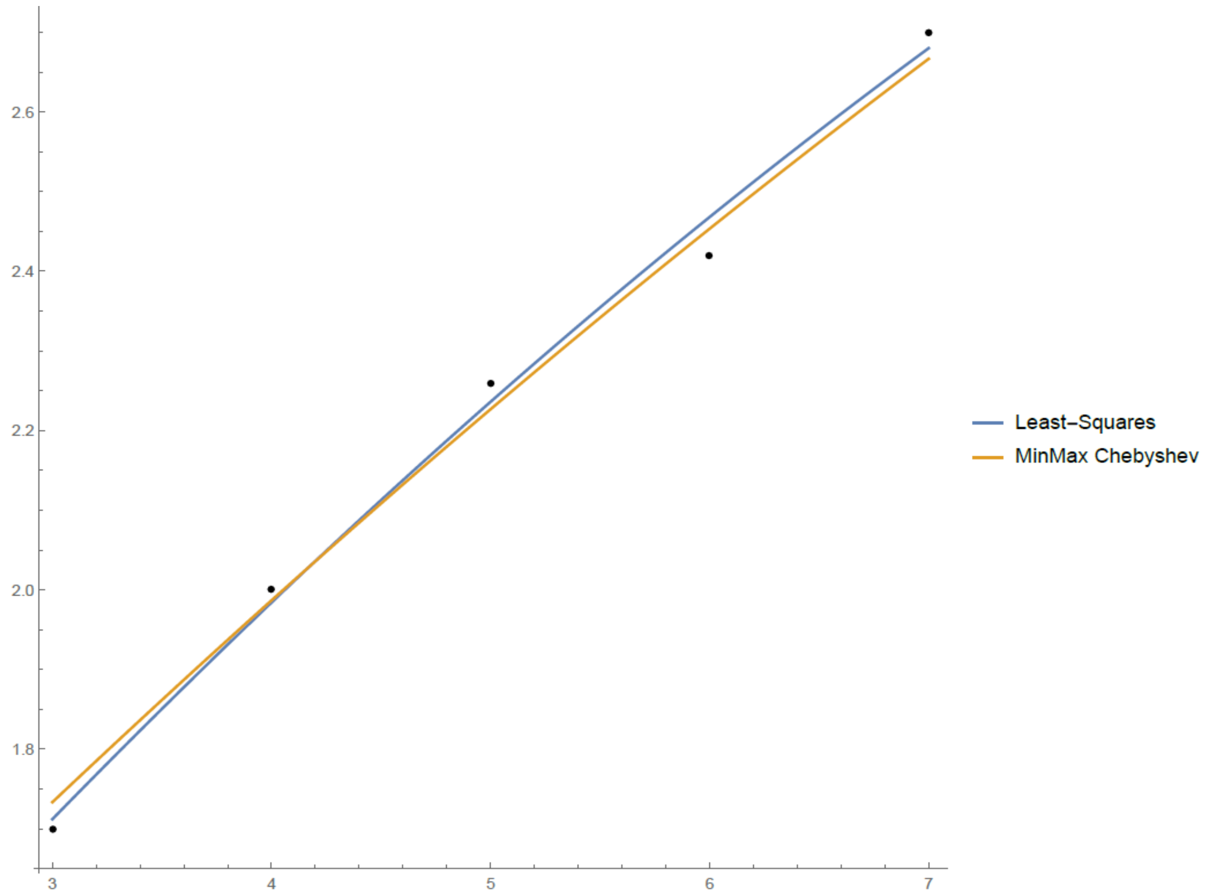

Figure 1.2: Parabolas approximating the 5 data points with x-arguments {3, 4, 5, 6, 7}. A possible set of basis functions is {1, x, x^2}. The parabola formula is $0.893333 + 0.300000\,x - 0.006667x^2$ for the minmax approximation (orange) and $0.776 + 0.342\,x - 0.01\,x^2$ for the least-squares approximation (blue), respectively.

1.1 Computational solution by a linear optimization problem

Mathematically, it is rather straightforward to transform the minmax objective (1.2a) into a linear optimization problem with constraints:

$$\begin{aligned} &\gamma \rightarrow \text{min! s.t. } r_i \le \gamma \wedge -r_i \le \gamma \quad (i = 0,\cdots,N) \quad \Leftrightarrow \\ &\gamma \rightarrow \text{min! s.t. } y_i - \sum_{j=0}^{m} a_j g_j(x_i) \le \gamma \quad \wedge \quad -y_i + \sum_{j=0}^{m} a_j g_j(x_i) \le \gamma \quad (i = 0,\cdots,N) \end{aligned} \tag{1.2b}$$

This says that the linear objective function γ has to minimized subject to the 2(N+1) linear inequality constraints denoted in (1.2b). The set of m+2 variables for the optimization is $\{a_0, a_1, \ldots, a_m, \gamma\}$.

Using the design matrix $G = \begin{pmatrix} g_0(x_0) & g_1(x_0) & \cdots & g_m(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ g_0(x_N) & g_1(x_N) & \cdots & g_m(x_N) \end{pmatrix}$ this can be written in a compact matrix form:

$$\begin{array}{l} \gamma \quad \to \quad \text{min! s.t.} \quad y| - G \cdot a| \le \gamma| \quad \wedge \quad -(y| - G \cdot a|) \le \gamma| \\ a| := (a_0, a_1, \ldots, a_m)^{tr}, \ \gamma| := (\gamma, \gamma, \ldots, \gamma)^{tr} \end{array} \tag{1.2c}$$

Only in exceptional cases can such a linear constrained optimization problem be solved analytically. Generally, the global solution has to be found by an iterative numerical scheme or by an advanced mathematical framework as cylindrical algebraic decomposition. The latter works even for algebraic functions and contraints and allows for exact solutions whenever the date is given as exact rational numbers. Another exact and global method is the Simplex algorithm (a standard used for linear optimization).

Example 1.2 (cont. Ex. 1.1)

The linear optimization problem (1.2c) for the data set in Example 1.1 has 10 inequalities as constraints. Below these are written out in matrix and expanded form, respectively:

$$\begin{pmatrix} 1.70 \\ 2.00 \\ 2.26 \\ 2.42 \\ 2.70 \end{pmatrix} - \overbrace{\begin{pmatrix} 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \\ 1 & 7 & 49 \end{pmatrix}}^{\text{Design matrix } G} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \le \begin{pmatrix} \gamma \\ \gamma \\ \gamma \\ \gamma \\ \gamma \end{pmatrix} \quad \wedge \quad - \begin{pmatrix} 1.70 \\ 2.00 \\ 2.26 \\ 2.42 \\ 2.70 \end{pmatrix} + \overbrace{\begin{pmatrix} 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \\ 1 & 7 & 49 \end{pmatrix}}^{\text{Design matrix } G} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} \le \begin{pmatrix} \gamma \\ \gamma \\ \gamma \\ \gamma \\ \gamma \end{pmatrix}$$

$$\begin{array}{l} -\text{a0} - 3\,\text{a1} - 9\,\text{a2} + 1.7 \le \gamma \\ -\text{a0} - 4\,\text{a1} - 16\,\text{a2} + 2. \le \gamma \\ -\text{a0} - 5\,\text{a1} - 25\,\text{a2} + 2.26 \le \gamma \\ -\text{a0} - 6\,\text{a1} - 36\,\text{a2} + 2.42 \le \gamma \\ -\text{a0} - 7\,\text{a1} - 49\,\text{a2} + 2.7 \le \gamma \\ \text{a0} + 3\,\text{a1} + 9\,\text{a2} - 1.7 \le \gamma \\ \text{a0} + 4\,\text{a1} + 16\,\text{a2} - 2. \le \gamma \\ \text{a0} + 5\,\text{a1} + 25\,\text{a2} - 2.26 \le \gamma \\ \text{a0} + 6\,\text{a1} + 36\,\text{a2} - 2.42 \le \gamma \\ \text{a0} + 7\,\text{a1} + 49\,\text{a2} - 2.7 \le \gamma \end{array}$$

The global solution to the constrained optimization problem $\gamma \quad \to \quad \text{min!}$ is unique:

```
{0.0333333, {γ → 0.0333333, a0 → 0.893333, a1 → 0.3, a2 → -0.00666667}}
```

Thus the parabola formula is 0.893333 + 0.300000 x − 0.00666667x^2 rounded to precision 6 (six significant digits).

2 Multi-variate linear minmax approximation

Multi-variate linear minmax problems are solved exactly along the same lines as uni-variate problems. The method developped in Sections 1.1 remains valid and keep the same form relying on the design matrix.

As in the multi-variate least-squares approximation the x-arguments now are vectors $\vec{x}_0, \vec{x}_1, \dots, \vec{x}_{n-1}, \vec{x}_N$ of dimension d and thus we need basis functions which take vectors as inputs (i.e. functions in several variables).

Example 1.3: Multi-variate least-squares for 4695 three-dimensional data points

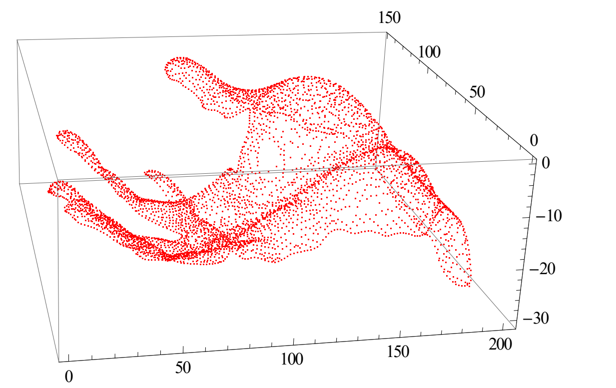

Figure 1.3a: 4695 data 3d-data points

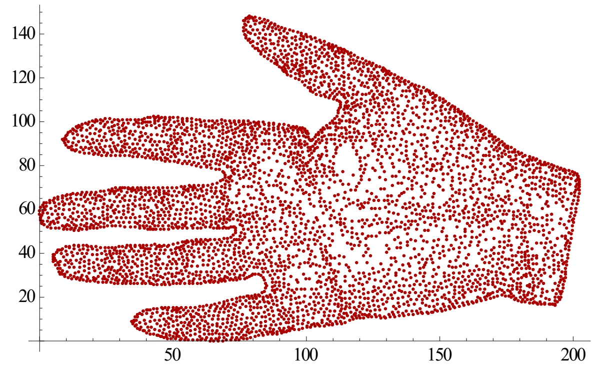

Figure 1.3b: Irregular grid in the argument plane (projections)

Degree 3 complete basis

Basis functions (not regularized): {1, x, y, x², 2xy, y², x³, 3x²y, 3xy², y³}

The first 4 data points are:

```
{ {82.0565,99.8271,-25.3321},{83.9503,99.6029,-24.8481},
  {85.8279,99.3811,-24.3083},{80.1541,99.9811,-25.753}, … }
```

The design matrix with respect to the statistically regularized basis $\left(\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^j \left(\frac{y-\mu_y}{\sigma_y}\right)^k \,\middle|\, j,k \in N_0\right\}\right)$ has dimensions 4695 x 10 and its first 3 rows are:

$$\begin{pmatrix} 1. & -0.320092 & 1.06937 & 0.102459 & -0.684594 & 1.14355 & -0.0327964 & 0.3287 & -1.09813 & 1.22288 \\ 1. & -0.283415 & 1.063 & 0.0803238 & -0.60254 & 1.12997 & -0.0227649 & 0.256153 & -0.960752 & 1.20116 \\ 1. & -0.24705 & 1.0567 & 0.0610339 & -0.522118 & 1.11662 & -0.0150785 & 0.193484 & -0.827585 & 1.17994 \end{pmatrix}$$

The system (1.2c) now reads as:

$$\gamma \;\to\; \min! \quad \text{s.t.} \;\; G\cdot a| \le \gamma| \;\;\wedge\;\; -G\cdot a| \le \gamma|$$

$$a| := \left(a_0, a_1, \dots, a_{m=9}\right)^{tr}, \;\; \gamma| := \Big(\underbrace{\gamma, \gamma, \dots, \gamma}_{N+1=4695}\Big)^{tr}$$

It consists of m + 2 = 11 variables $\{a_0, a_1, \ldots, a_{m=9}, \gamma\}$ and has 2(N + 1) = 2 * 4695 linear inequality constraints. Below two small sets of these are written out (notice that the variables are re-numbered beginning with index 1):

$$-25.3321 - 1.\ a1 - 1.22288\ a10 + 0.320092\ a2 - 1.06937\ a3 - 0.102459\ a4 + 0.684594\ a5 - 1.14355\ a6 + 0.0327964\ a7 - 0.3287\ a8 + 1.09813\ a9 \le \gamma$$
$$-24.8481 - 1.\ a1 - 1.20116\ a10 + 0.283415\ a2 - 1.063\ a3 - 0.0803238\ a4 + 0.60254\ a5 - 1.12997\ a6 + 0.0227649\ a7 - 0.256153\ a8 + 0.960752\ a9 \le \gamma$$
$$-24.3083 - 1.\ a1 - 1.17994\ a10 + 0.24705\ a2 - 1.0567\ a3 - 0.0610339\ a4 + 0.522118\ a5 - 1.11662\ a6 + 0.0150785\ a7 - 0.193484\ a8 + 0.827585\ a9 \le \gamma$$

$$25.3321 + 1.\ a1 + 1.22288\ a10 - 0.320092\ a2 + 1.06937\ a3 + 0.102459\ a4 - 0.684594\ a5 + 1.14355\ a6 - 0.0327964\ a7 + 0.3287\ a8 - 1.09813\ a9 \le \gamma$$
$$24.8481 + 1.\ a1 + 1.20116\ a10 - 0.283415\ a2 + 1.063\ a3 + 0.0803238\ a4 - 0.60254\ a5 + 1.12997\ a6 - 0.0227649\ a7 + 0.256153\ a8 - 0.960752\ a9 \le \gamma$$
$$24.3083 + 1.\ a1 + 1.17994\ a10 - 0.24705\ a2 + 1.0567\ a3 + 0.0610339\ a4 - 0.522118\ a5 + 1.11662\ a6 - 0.0150785\ a7 + 0.193484\ a8 - 0.827585\ a9 \le \gamma$$

Minmax approximation as a variable list and a functional formula, respectively:

$$\{\gamma \to 7.74506,\ a1 \to -16.0839,\ a2 \to 18.4127,\ a3 \to -0.55464,\ a4 \to -0.121244,\ a5 \to 0.0184736,\ a6 \to 2.09166,\ a7 \to -5.84413,\ a8 \to -0.0488747,\ a9 \to -2.00871,\ a10 \to 0.0229543\}$$

$$-16.0839 + 0.356605\,(-98.584 + x) - 0.0000454777\,(-98.584 + x)^2 - 0.0000424549\,(-98.584 + x)^3 - 0.0157519\,(-62.1736 + y) + 0.0000203224\,(-98.584 + x)\,(-62.1736 + y) - 1.56195 \times 10^{-6}\,(-98.584 + x)^2\,(-62.1736 + y) + 0.00168708\,(-62.1736 + y)^2 - 0.0000941355\,(-98.584 + x)\,(-62.1736 + y)^2 + 5.25814 \times 10^{-7}\,(-62.1736 + y)^3$$

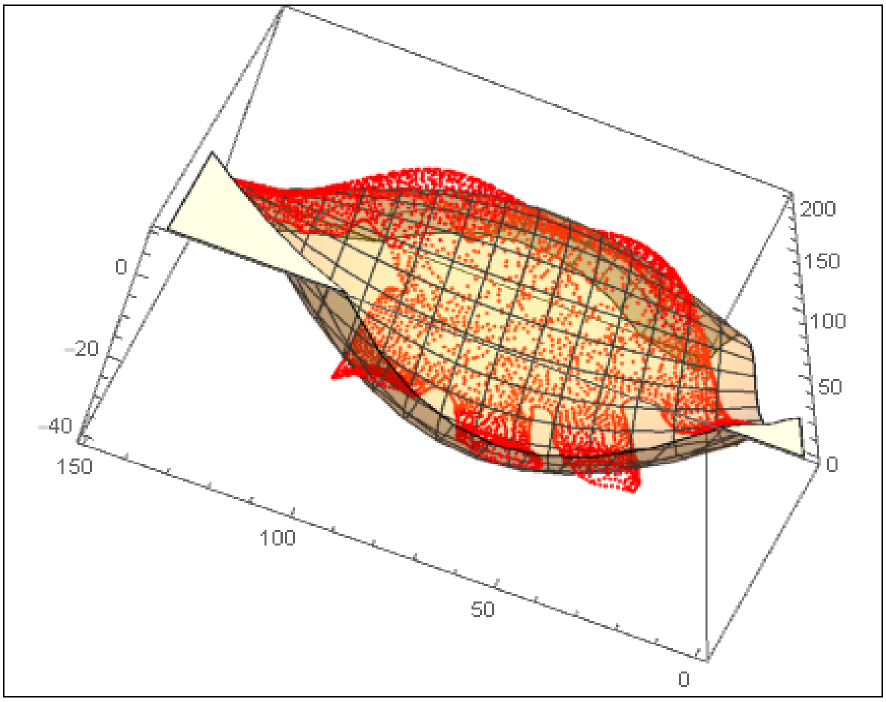

Figure 1.4a: 3th total degree minmax approximation function and data points (red).

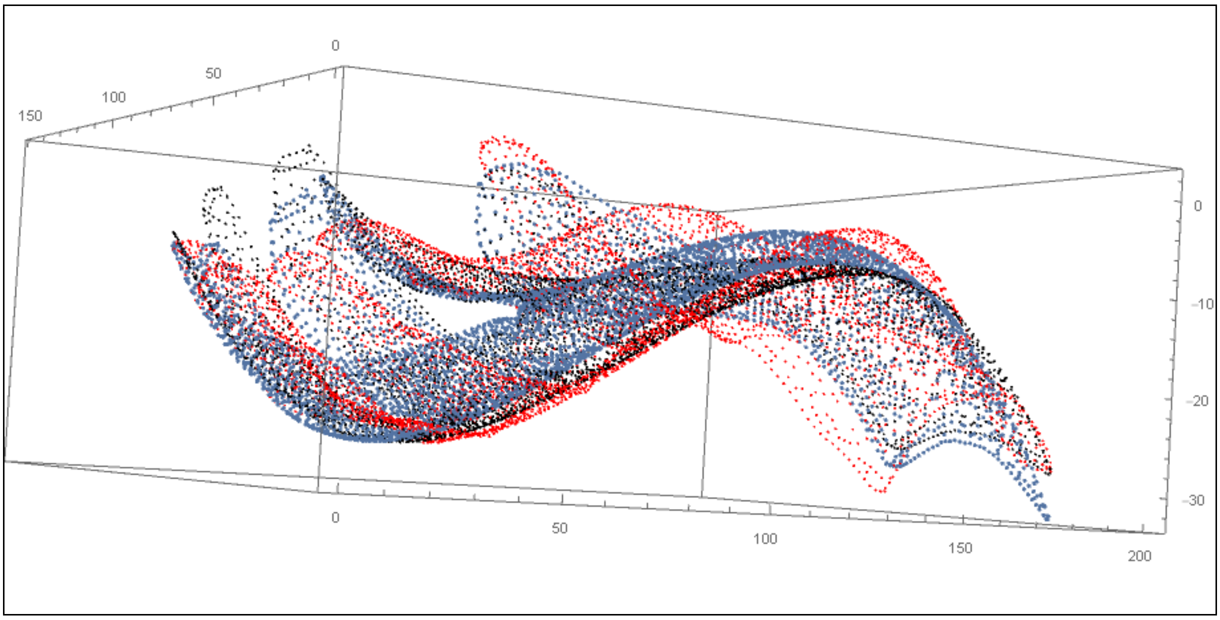

Figure 1.4b: Scatter plots of 3th total degree minmax approximation (blue) compared to least-squares approximation (black) and data points (red).