1.1.3 Van der Pol's Limit Cycle

In order to be able to set a limit to thought, we should have to find both sides of the limit thinkable (i.e., we should have to be able to think what cannot be thought).
Ludwig Wittgenstein, Austrian philosopher (1889–1951)

In the first two recipes, the ODEs were linear and, because they had constant coefficients, were easily solved analytically. In this recipe, we look at the historically important nonlinear *Van der Pol* ODE, for which no analytic solution exists. Balthasar Van der Pol was a Dutch electrical engineer who pioneered the development of experimental nonlinear dynamics in the 1920s and 1930s using electrical circuits and discovered several important nonlinear phenomena.

For example, he found that certain nonlinear circuits containing vacuum tubes could begin to spontaneously oscillate even though the energy source was constant, the oscillations evolving into a stable cycle, now called a *limit cycle*. When these circuits were driven with a signal whose frequency was near that of the limit cycle, the resulting periodic response shifted its frequency to that of the driving signal. The circuit became *entrained* to the driving signal. Entrainment is the basis of the modern pacemaker, which is used to stabilize irregular heart beats, or *arrhythmias*.

In the September 1927 issue of the journal *Nature*, Van der Pol and van der Mark reported that an "irregular noise" was heard at certain driving frequencies, probably one of the first experimental reports of *deterministic chaos*.[4]

Here, we shall look at a modern electrical circuit [Cho64] that is governed by the Van der Pol equation and can produce his limit cycle. The circuit, involving a battery (voltage V_B), inductor L, resistor R, capacitor C, and a tunnel diode D, is shown on the left of Figure 1.10. The tunnel diode has a *nonlinear* current (i_D)-voltage (V_D) curve similar to that shown on the right.

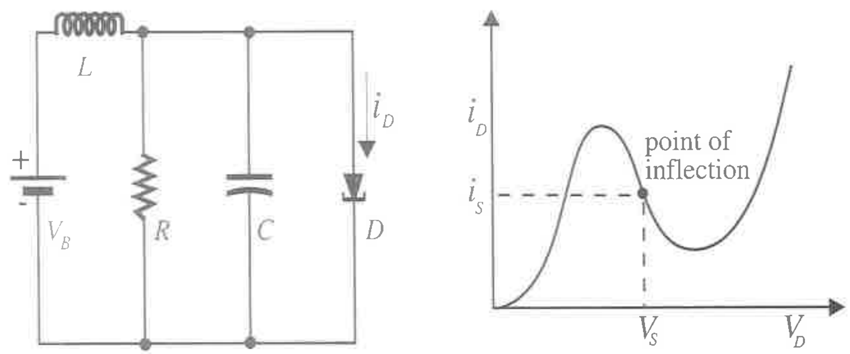

Figure 1.10: Left: tunnel diode circuit. Right: Current-voltage curve for diode.

[4]This historical information is taken from the IEEE History Center (www.ieee.org).

The battery voltage V_B is adjusted to coincide with the inflection point V_S of the i_D vs. V_D curve, i.e., $V_B = V_S$. Near this operating point, one may write $i = -a\,v + b\,v^3$, where $i = i_D - i_S$ and $v = V_D - V_S$, and a and b are positive.

The governing Van der Pol (VdP) ODE, which will presently be derived, is

$$\ddot{x} - \epsilon\,(1 - x^2)\,\dot{x} + x = 0, \quad \epsilon > 0, \tag{1.8}$$

with x proportional to v. Equation (1.8) is just the simple harmonic oscillator equation for unit frequency and mass with an amplitude-dependent damping term. For $x < 1$, the damping contribution is negative, so that oscillations tend to grow, while for $x > 1$ the damping is positive, tending to reduce the oscillations. The negative damping is responsible for the growth of any small spontaneous circuit "noise" into stable oscillations, i.e., into a stable limit cycle.

Let's now derive the VdP equation and demonstrate the growth of a small input signal into a limit cycle for a typical tunnel diode, 1N3719, for which $a = 0.05$ and $b = 1.0$ in SI units.

The DEtools and PDEtools packages are loaded. The former contains the DEplot3d command, which is a three-dimensional generalization of the DEplot command. The PDEtools package contains the dchange command, which will allow us to easily make a somewhat complicated variable transformation.

```
> restart: with(DEtools): with(PDEtools):
```

The time-dependent tunnel diode current and voltage expressions are entered.

```
> i[D]:=i[S]-a*v(t)+b*v(t)^3; V[D]:=V[S]+v(t);
```

$$i_D := i_S - a\,v(t) + b\,v(t)^3 \qquad V_D := V_S + v(t)$$

The voltage drop across both the resistor R and the capacitor C is the same as across the diode D, i.e., $V_R = V_D$ and $V_C = V_D$. The voltage drop across the inductor L is $V_L = V_B - V_D = V_S - V_D$, the latter form being entered.

```
> V[R]:=V[D]: V[C]:=V[D]: V[L]:=V[S]-V[D];
```

$$V_L := -v(t)$$

By *Ohm's law*, the current through the resistor is $i_R = V_R/R$. The current through the capacitor is $i_C = C\,(dV_C/dt)$.

```
> i[R]:=V[R]/R; i[C]:=C*diff(V[C],t);
```

$$i_R := \frac{V_S + v(t)}{R} \qquad i_C := C\left(\frac{d}{dt}\,v(t)\right)$$

Using *Kirchhoff's current rule*, *eq1* states that the current leaving L must be equal to the sum of the currents entering R, C, and D.

```
> eq1:=-i[L](t)+i[R]+i[C]+i[D]=0; #Kirchhoff's current rule
```

$$eq1 := -i_L(t) + \frac{V_S + v(t)}{R} + C\left(\frac{d}{dt}\,v(t)\right) + i_S - a\,v(t) + b\,v(t)^3 = 0$$

Differentiating *eq1* with respect to t eliminates the inflection point current i_S.

```
> eq2:=expand(diff(eq1,t)/C);
```

$$eq2 := \frac{\frac{d}{dt}v(t)}{C\,R} - \frac{\frac{d}{dt}i_L(t)}{C} + \left(\frac{d^2}{dt^2}v(t)\right) - \frac{a\left(\frac{d}{dt}v(t)\right)}{C} + \frac{3\,b\,v(t)^2\left(\frac{d}{dt}v(t)\right)}{C} = 0$$

From the definition of inductance, one has $di_L/dt = V_L/L$, which is substituted into *eq2*. This yields a second-order ODE entirely in terms of the potential $v(t)$.

```
> de:=subs(diff(i[L](t),t)=V[L]/L,eq2);
```

$$de := \frac{v(t)}{C\,L} + \frac{\frac{d}{dt}v(t)}{C\,R} + \left(\frac{d^2}{dt^2}v(t)\right) - \frac{a\left(\frac{d}{dt}v(t)\right)}{C} + \frac{3\,b\,v(t)^2\left(\frac{d}{dt}v(t)\right)}{C} = 0$$

Then *de* is put into more compact form by collecting the first derivatives. The resulting ODE is the *unnormalized* form of the VdP equation.

```
> de1:=collect(de,diff(v(t),t)); #unnormalized VdP equation
```

$$de1 := \left(\frac{1}{C\,R} - \frac{a}{C} + \frac{3\,b\,v(t)^2}{C}\right)\left(\frac{d}{dt}v(t)\right) + \frac{v(t)}{C\,L} + \left(\frac{d^2}{dt^2}v(t)\right) = 0$$

To obtain the *normalized* (dimensionless) VdP equation, a transformation will be made to new variables. First a characteristic frequency $\omega = 1/\sqrt{L\,C}$ is introduced by making the following substitution into *de1*.

```
> de2:=subs(v(t)/(C*L)=omega^2*v(t),de1);
```

$$de2 := \left(\frac{1}{C\,R} - \frac{a}{C} + \frac{3\,b\,v(t)^2}{C}\right)\left(\frac{d}{dt}v(t)\right) + \omega^2\,v(t) + \left(\frac{d^2}{dt^2}v(t)\right) = 0$$

Inspecting the structure of *de2*, we are led to introduce a dimensionless time $\tau = \omega\,t$, and voltage $x(\tau) = \sqrt{(3\,b)}\,v(t)/\sqrt{(a - 1/R)}$. The transformation from the "old" $(t, v(t))$ to the "new" $(\tau, x(\tau))$ variables is entered.

```
> tr:={t=tau/omega,v(t)=x(tau)*sqrt(a-1/R)/sqrt(3*b)}:
```

The dchange command allows us to apply the transformation to *de2*. The result is then multiplied by the factor $\sqrt{(3\,b)}/(\omega^2\,\sqrt{(a - 1/R)})$.

```
> sqrt(3*b)*dchange(tr,de2,[x(tau),tau])/(omega^2*sqrt(a-1/R)):
```

Using the ditto operator, %, to refer[5] to the last computed result, we collect $dx(\tau)/d\tau$ terms and factor the result.

```
> de3:=collect(%,diff(x(tau),tau),factor);
```

$$de3 := \frac{(x(\tau) - 1)\,(x(\tau) + 1)\,(a\,R - 1)\left(\frac{d}{d\tau}x(\tau)\right)}{\omega\,C\,R} + x(\tau) + \left(\frac{d^2}{d\tau^2}x(\tau)\right) = 0$$

Introducing the dimensionless parameter $\epsilon = (a\,R - 1)/(\omega\,C\,R)$, the normalized Van der Pol equation results.

```
> vdp:=subs((a*R-1)=epsilon*(omega*C*R),de3); #VdP equation
```

$$vdp := (x(\tau) - 1)\,(x(\tau) + 1)\,\epsilon\left(\frac{d}{d\tau}x(\tau)\right) + x(\tau) + \left(\frac{d^2}{d\tau^2}x(\tau)\right) = 0$$

To make a phase-plane picture, the second-order VdP equation is now rewritten as two first-order ODEs in *de4* and *de5*, by setting $y(\tau) \equiv dx(\tau)/d\tau$.

```
> de4:=diff(x(tau),tau)=y(tau); de5:=subs(de4,vdp);
```

[5] To refer to the second to last computed result, use %%.

$$de4 := \frac{d}{d\tau}x(\tau) = y(\tau)$$

$$de5 := (x(\tau) - 1)\,(x(\tau) + 1)\,\epsilon\,y(\tau) + x(\tau) + \left(\frac{d}{d\tau}y(\tau)\right) = 0$$

With $a = 0.05$ entered for the tunnel diode 1N3719, a necessary condition for a limit cycle to occur is that $\epsilon > 0$ or $R > 1/a = 1/0.05 = 20$ ohms. We take $R = 55$ ohms, $L = 25.0 \times 10^{-3}$ henries, and $C = 10^{-6}$ farads, and calculate ω and ϵ.

```
> a:=0.05: R:=55: L:=25.0*10^(-3): C:=10^(-6):
> omega:=1/sqrt(L*C); epsilon:=(a*R-1)/(omega*C*R);
```

$$\omega := 6324.555320 \qquad \epsilon := 5.030896278$$

The VdP equation has a fixed point at the origin of the y vs. x phase plane. Let's choose an initial condition close to this point, viz., $x(0) = 0.1$, $y(0) = 0$.

```
> ic:=x(0)=0.1,y(0)=0:
```

Instead of plotting the trajectory in two dimensions using either the DEplot or phaseportrait commands, the solution curve corresponding to the initial condition can be drawn in the three-dimensional τ vs. x vs. y space using DEplot3d with the option scene=[tau,x,y]. The line color of the trajectory is allowed to vary with τ. The resulting trajectory appears in a 3-dimensional viewing box similar to that shown in Figure 1.11. The viewing box can be rotated on the computer screen, by clicking on the box and dragging with the

```
> DEplot3d([de4,de5],[x(tau),y(tau)],tau=0..60,
  scene=[tau,x,y],[[ic]],stepsize=0.01,linecolor=tau);
```

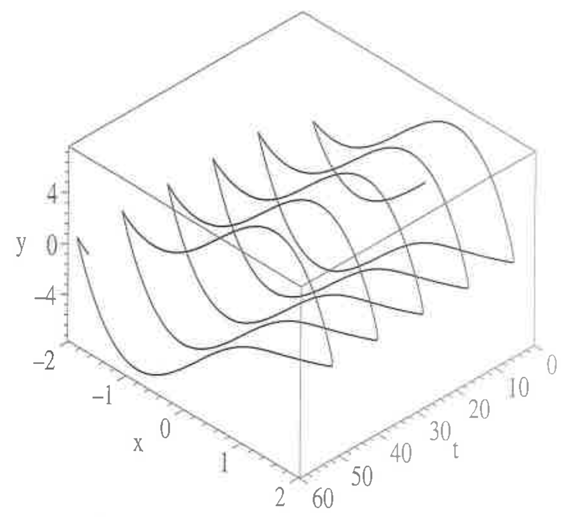

Figure 1.11: Evolution of the VdP trajectory onto a limit cycle.

mouse. The angular coordinates, θ and ϕ, of the viewing box appear in a small window near the top left of the computer screen, the default angles being $45°, 45°$. The option `orientation=[angle,angle]`, with the values of the two angles specified, can be inserted into `DEplot3d` if some other default orientation is desired. For example, choose the angles to be $\theta = 0$, $\phi = 90$ to see the phase plane.

The trajectory evolves away from the vicinity of the origin onto a closed loop, the limit cycle. You can check that the limit cycle will be obtained no matter what the choice of initial condition.[6] Can you identify the stationary point at the origin?

The nonlinear Van der Pol equation does not have an analytic solution. Applying the `dsolve` command to the ODE system, *de4* and *de5*, subject to the initial condition,

```
> dsolve({de4,de5,ic},{x(tau),y(tau)});
```

produces no output for $x(t)$ and $y(t)$. Only a few nonlinear ODEs of physical interest have analytic solutions. We shall see a few examples in Chapter 4.

PROBLEMS:

Problem 1-6: Different initial conditions

With all other parameters as in the text recipe, show for a number of different initial conditions that all trajectories wind onto the limit cycle. Take the orientation that shows the y versus x phase plane.

Problem 1-7: Varying the resistance

With all other parameters as in the text recipe, investigate the behavior of the Van der Pol equation as the resistance R is varied. Choose an orientation that shows x versus τ. Discuss the results.

Problem 1-8: Tangent field

In the text recipe, use the `phaseportrait` command instead of `DEplot3d` to make a phase-plane portrait with the tangent field included.

1.2 Three-Dimensional Autonomous Systems

Although a three-dimensional plot was produced in the last example, we were still dealing with a two-dimensional autonomous system. Let's now consider a general three-dimensional autonomous ODE system of the structure

$$\dot{X} = P(X,Y,Z), \quad \dot{Y} = Q(X,Y,Z), \quad \dot{Z} = R(X,Y,Z), \qquad (1.9)$$

with P, Q, and R known functions of the three dependent variables X, Y, and Z. Some systems of physical interest are naturally of this structure, while the two-dimensional nonautonomous ODE system

$$\dot{X} = P(X,Y,t), \quad \dot{Y} = Q(X,Y,t), \qquad (1.10)$$

can be recast into the form (1.9) by setting $R = 1$ and imposing $Z(0) = 0$.

1.2.1 The Period-Doubling Route to Chaos

Chaos often breeds life, when order breeds habit.
Henry Adams, American historian (1838–1918)

Nonautonomous nonlinear ODEs, such as *Duffing's equation*,

$$\ddot{x} + 2\gamma\dot{x} + \alpha x + \beta x^3 = F\cos(\omega t), \qquad (1.11)$$

have played a very important role in the development of nonlinear dynamics. Duffing's equation is a model for the motion of a viscously damped (damping coefficient γ) spring that is subject to a nonlinear restoring force $f = -\alpha x - \beta x^3$ and is being driven by a periodic force of amplitude F and frequency ω. Depending on the signs and magnitudes of α and β, various descriptive names are usually applied to Duffing's equation:

- hard-spring Duffing equation: $\alpha > 0$, $\beta > 0$;
- soft-spring Duffing equation: $\alpha > 0$, $\beta < 0$;
- inverted Duffing equation: $\alpha < 0$, $\beta > 0$;
- nonharmonic Duffing equation: $\alpha = 0$, $\beta > 0$.

Setting $\dot{x} = y$, Duffing's equation can be written in the 2-dimensional nonautonomous form (1.10) with $P \equiv y$ and $Q \equiv -2\gamma y - \alpha x - \beta x^3 + F\cos(\omega t)$. It can be made autonomous by introducing a third dependent variable, z, and expressing Duffing's equation as the three-dimensional system

$$\dot{x} = y, \quad \dot{y} = -2\gamma y - \alpha x - \beta x^3 + F\cos(z), \quad \dot{z} = \omega, \text{ with } z(0) = 0. \qquad (1.12)$$

After a transient time interval, the Duffing system can, not unexpectedly, display a periodic oscillation in response to the periodic driving term. A more surprising result is that it can exhibit highly irregular, or *chaotic*, oscillatory motion that is essentially unpredictable, even though the Duffing equation is deterministic. In contrast to the periodic regime, there is an extreme sensitivity to initial conditions in the chaotic domain.

The Duffing ODE is not the only dynamical system to exhibit chaotic behavior. In general, for chaos to occur in a dynamical system, two ingredients are necessary, namely that some nonlinearity be present and that the system have at least three dynamical dependent variables (i.e., be at least three-dimensional). The study of chaotic behavior is a nonlinearly growing field, and it is not our intention to explore it in any depth in this text, although some useful diagnostic tools are briefly presented in the Desserts.

In the following recipe, Jennifer, a mathematician at MIT, will illustrate the so-called *period-doubling route to chaos* for the Duffing system. This refers to a sequence of period doublings (halving of the frequency response) that are observed when a "control" parameter is increased, ultimately ending in a chaotic regime. This period-doubling scenario is not the only route to chaos, but it is a very common one in the study of driven nonlinear ODE systems as well as other